(where a, b, c, and k are constants). Find a vector field whose flow is the set of solutions to this system.

Solution If we think of $(r(t), w(t))$ as a point in the phase plane then $r'(t)\vec{i} + w'(t)\vec{j}$ is a velocity vector, and the system gives a vector field

$$\vec{v} = (aw + crw)\vec{i} + (-br + krw)\vec{j}.$$

Notice that the vector field obtained in the last example contains more information than the slope fields sometimes used to study first order differential equations. In particular, the arrowheads of the vector field show the direction of the flow. If we draw slope fields for systems, we do so by eliminating t, so the direction of the trajectories is lost.

The Divergence and Volume Change

We have seen how to think of any vector field in terms of a flow. We will exploit this interpretation to arrive at a measure of compression or expansion of the fluid. This measure is called the *divergence* of a vector field, and has significant application in the theory of fluids, heat flow, and electricity and magnetism. Most instances in which the divergence is applied to a vector field will be in 3-space, but for simplicity we shall explore the concept in two-dimensional vector fields — the results carry over to three dimensions with little change. Assume that $\vec{F}$ is a velocity field and imagine that it describes the flow of a gas, such as air, that can expand and contract as it flows. To see the expansion or contraction, we follow a small area of gas as it moves. See Figure 17.29.

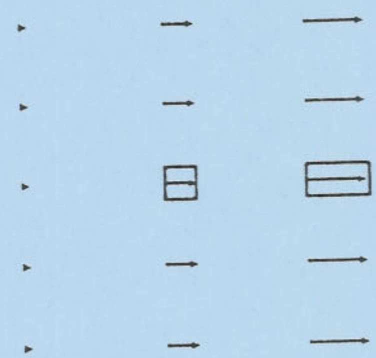

Figure 17.29: A moving volume of gas at two times, showing expansion

Example 5 Determine whether the gas in the regions depicted in the velocity fields of Figure 17.30 is expanding or contracting.

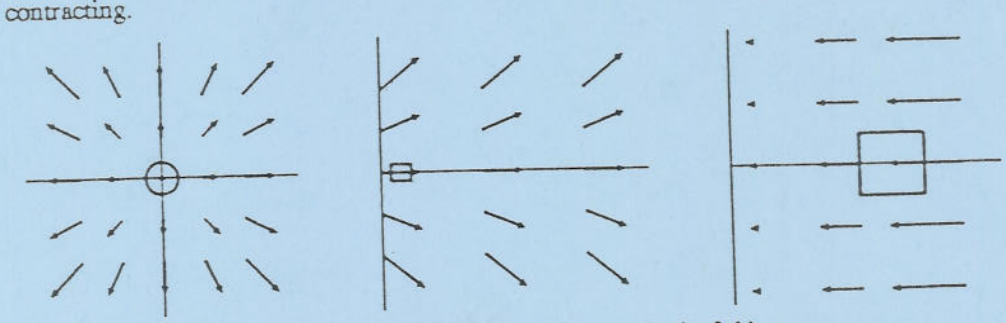

Figure 17.30: Three regions in three velocity fields

Solution In Figure 17.31 we have sketched the regions the gas will occupy after flowing along with the velocity fields for a short time. In the first two diagrams, the gas occupies a larger volume than originally, so it is expanding. The third diagram shows a shrinking volume, so the gas is contracting.

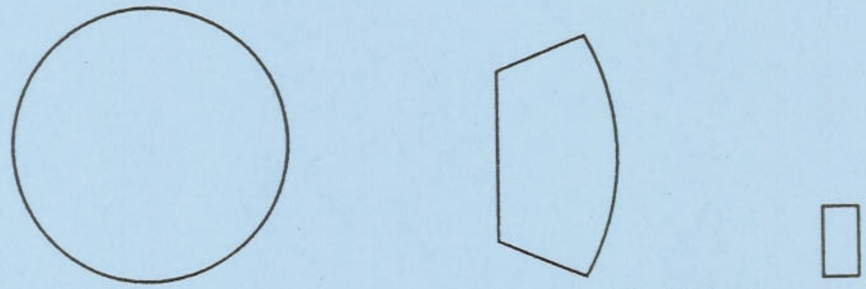

Figure 17.31: The three regions in Figure 17.30 a short period of time later

Now we will measure the rate a vector field is expanding. Suppose we want to calculate the rate of expansion described by a velocity vector field

$$\vec{v} = a(x, y)\vec{i} + b(x, y)\vec{j}.$$

Since the rate may vary from point to point, we will fix a point $P = (x_0, y_0)$, and compute the rate at P. We focus on the gas at time t_0 in a small box of dimensions $\Delta x \times \Delta y$ with edges parallel to the axes and with one corner at P, as shown in Figure 17.32. Now imagine the small box and the gas inside moving according to the flow of the velocity field. The rectangle will change shape (and area) if the gas is getting compressed or expanding. It is the rate of change of the area of this small box that we shall now estimate. We let $A(t)$ denote the area of the small box at time t and we will compute

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{A(t_0 + \Delta t) - A(t_0)}{\Delta t}.$$

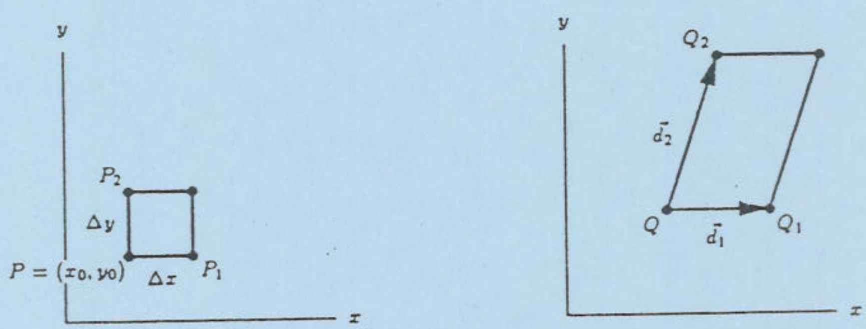

Figure 17.32: A small box at time t_0 and at $t_0 + \Delta t$, a short period of time later

At time t_0, the area is given by

$$A(t_0) = \Delta x \Delta y.$$

By time $t_0 + \Delta t$, the flow will have moved the gas from the box to a new region whose area $A(t_0 + \Delta t)$ we wish to approximate. To find the corners of the new region, we begin at the corners of the original box and move with velocity $\vec{v}$ for time Δt. The point P will flow (approximately) to the point

$$Q = P + \vec{v}(P)\Delta t = (x_0, y_0) + (a(P)\vec{i} + b(P)\vec{j})\Delta t.$$

Similarly, the point $P_1 = (x_0 + \Delta x, y_0)$ will flow (approximately) to

$$Q_1 = P_1 + \vec{v}(P_1)\Delta t.$$

Using the linear approximation

$$\vec{v}(P_1) = a(x_0 + \Delta x, y_0)\vec{i} + b(x_0 + \Delta x, y_0)\vec{j} \approx (a(P) + a_x(P)\Delta x)\vec{i} + (b(P) + b_x(P)\Delta x)\vec{j},$$

we find that

$$Q_1 \approx (x_0 + \Delta x, y_0) + (a(P) + a_x(P)\Delta x)\vec{i} + (b(P) + b_x(P)\Delta x)\vec{j}.$$

The edge of the region from Q to Q_1 can be described (approximately) by the displacement vector

$$\vec{d}_1 = (\Delta x + a_x\Delta x\Delta t)\vec{i} + b_x\Delta x\Delta t\vec{j},$$

where a_x, b_x are evaluated at P. A similar computation shows the displacement vector for the edge QQ_2 is approximated by

$$\vec{d}_2 = a_y\Delta y\Delta t\vec{i} + (\Delta y + b_y\Delta y\Delta t)\vec{j},$$

where a_y, b_y are evaluated at P. See Figure 17.32.

The region occupied by the gas at time $t_0 + \Delta t$, where Δt is a short time, will be approximately a parallelogram with sides $\vec{d}_1$ and $\vec{d}_2$. Now the area of a parallelogram [2] with sides given by the vectors $u_1\vec{i} + u_2\vec{j}$ and $v_1\vec{i} + v_2\vec{j}$ is $u_1v_2 - u_2v_1$. So the area of the parallelogram is

$$\begin{aligned}A(t_0 + \Delta t) &\approx (\Delta x + a_x\Delta x\Delta t)(\Delta y + b_y\Delta y\Delta t) - (b_x\Delta x\Delta t)(a_y\Delta y\Delta t)\\ &= \Delta x\Delta y[(1 + a_x\Delta t)(1 + b_y\Delta t) - (b_x\Delta t)(a_y\Delta t)]\\ &= \Delta x\Delta y[1 + a_x\Delta t + b_y\Delta t + (a_xb_y - b_xa_y)(\Delta t)^2]\end{aligned}$$

Finally, we can compute the difference quotient for dA/dt:

$$\begin{aligned}A'(t_0) &= \lim_{\Delta t \to 0} \frac{(A(t_0 + \Delta t) - A(t_0))}{\Delta t}\\ &\approx \lim_{\Delta t \to 0} \frac{\Delta x\Delta y(1 + a_x\Delta t + b_y\Delta t + (a_xb_y - b_xa_y)(\Delta t)^2) - (\Delta x\Delta y)}{\Delta t}\\ &= \lim_{\Delta t \to 0} \Delta x\Delta y(a_x + b_y + (a_xb_y - b_xa_y)\Delta t).\end{aligned}$$

Thus, as $\Delta t \to 0$, we see that the rate of change of the area or the rectangle is given by

$$\frac{dA}{dt} = \Delta x\Delta y(a_x + b_y).$$

To get a result that is independent of the area of the original rectangle at P, we compute the rate of change of area *per unit area*, that is $A'(t_0)/A(t_0)$, getting

$$\frac{A'(t_0)}{A(t_0)} \approx a_x + b_y$$

[2] This is a *signed* area, which is what we want.

Solution The field represents the flow horizontally outward from a source along the y-axis. The flow of this vector field consists of horizontal lines, with speed increasing as distance from the y-axis increases. If we place a small box at a point P in the flow in Figure 17.34, then in time Δt the box will become elongated due to the higher velocity at its right end. Thus, there is a positive rate of change of area (that is, fluid is expanding), and we expect the divergence to be positive. To confirm, compute the divergence exactly as

$$\operatorname{div}\vec{v} = a_x + b_y = \frac{\partial x}{\partial x} + \frac{\partial 0}{\partial y} = 1.$$

This tells us that the rate of expansion is constant throughout the flow.

Example 7 The velocity field $\vec{v} = x\vec{j}$ is shown in Figure 17.35. Discuss the flow and divergence of this field.

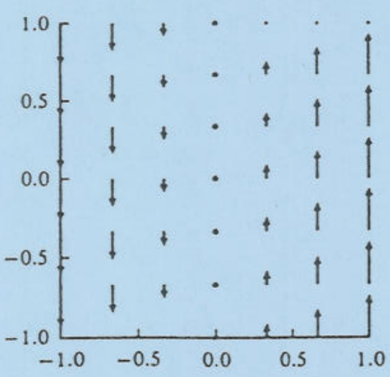

Figure 17.35: The velocity field $\vec{v} = x\vec{j}$

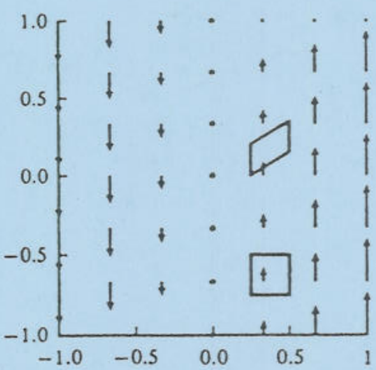

Figure 17.36: Motion of a small volume with the vector field $\vec{v} = x\vec{j}$

Solution The field in Figure 17.35 represents a flow parallel to the y-axis, with no apparent source (this kind of flow is called a *shear*). The flow of this vector field consists of vertical lines, with constant speed along the lines. If we place a small box at a point P in the flow, then in time Δt the box will become a parallelogram due to the higher velocity at its right end and lower velocity at its left end (See Figure 17.36.) However, this parallelogram has the same area as the original box, so it appears there is a zero rate of change of area. If we compute the divergence exactly we see that

$$\operatorname{div}\vec{v} = \frac{\partial 0}{\partial x} + \frac{\partial x}{\partial y} = 0.$$

This tells us that there is no expansion or contraction anywhere in this flow.

Problems for Section 17.2

Sketch the vector field and flow for the vector fields in Problems 1 - 3.

1. $\vec{v} = 3\vec{i}$
2. $\vec{v} = 2\vec{j}$
3. $\vec{v} = 3\vec{i} - 2\vec{j}$