7.2 #2c One can observe that $\vec{F} = (yz, xz, xy)$ is a gradient v.f., $\vec{F} = \vec{\nabla} f$ for $f(x,y,z) = xyz$. Then

$$\int_C \vec{F} \cdot d\vec{s} = f(0,0,1) - f(1,0,0)$$
$$= 0 - 0 = 0$$

Alternatively we can integrate piece by piece; take $C = C_1 + C_2$.

Say C_1 runs from $(1,0,0)$ in a straight line to $(0,1,0)$. Then the direction vector is $(-1,1,0)$ and we have a parameterization

$$\begin{cases} x = 1 - t \\ y = t \\ z = 0 \end{cases} \quad \text{for } 0 \leq t \leq 1. \text{ Now}$$

we compute $dx = -dt$, $dy = dt$, $dz = 0dt$, which leads to

$$\int_{C_1} yz\,dx + xz\,dy + xy\,dz = \int_0^1 0dt = 0.$$

For C_2 from $(0,1,0)$ to $(0,0,1)$ we have $x = 0$, $y = 1 - t$, $z = t$ for $0 \leq t \leq 1$, and $dx = 0dt$, $dy = -dt$, $dz = t$; hence

$$\int_{C_2} yz\,dx + xz\,dy + xy\,dz = \int_0^1 0dt = 0.$$

7.2 #5 First observe that $\left| \int_C \vec{F} \cdot d\vec{s} \right| \leq \int_C \left| \vec{F} \cdot d\vec{s} \right|$. For the moment we will pretend this makes sense.

On the left hand side there may be parts of C where $\vec{F} \cdot d\vec{s} > 0$ and others where $\vec{F} \cdot d\vec{s} < 0$; these will cancel out before you even take absolute value of the integral. On the right everything is made positive before integrating. Now

$$\left| \vec{F} \cdot d\vec{s} \right| = \left| \|\vec{F}\| \|d\vec{s}\| \cos\theta \right|$$

$\leq \|\vec{F}\| \|d\vec{s}\|$ since $|\cos\theta| \leq 1$.

Now $\|d\vec{s}\| = ds$ and $\|\vec{F}\| \leq M$, so our integral is $\leq M \int_C ds = Ml$.

This is the gist. A bit more precisely,

$$\left| \int_C \vec{F} \cdot d\vec{s} \right| \overset{\text{def'n}}{=} \left| \int_C \vec{F} \cdot \vec{c}\,'(t)\, dt \right|$$

for the reason discussed above $\quad \leq \int_C \left| \vec{F} \cdot \vec{c}\,'(t) \right| dt$ Not inside absval.

θ the angle between $\vec{F}$ and $\vec{c}\,'(t)$ at $\vec{c}(t)$ where $\vec{F}$ and $\vec{c}\,'(t)$ are rooted

$$= \int_C \|\vec{F}\| \|\vec{c}\,'(t)\| |\cos\theta|\, dt$$

$$\leq \int_C M \|\vec{c}\,'(t)\|\, dt \quad \text{since } |\cos\theta| \leq 1$$

$$= \int_C M\, ds = Ml$$

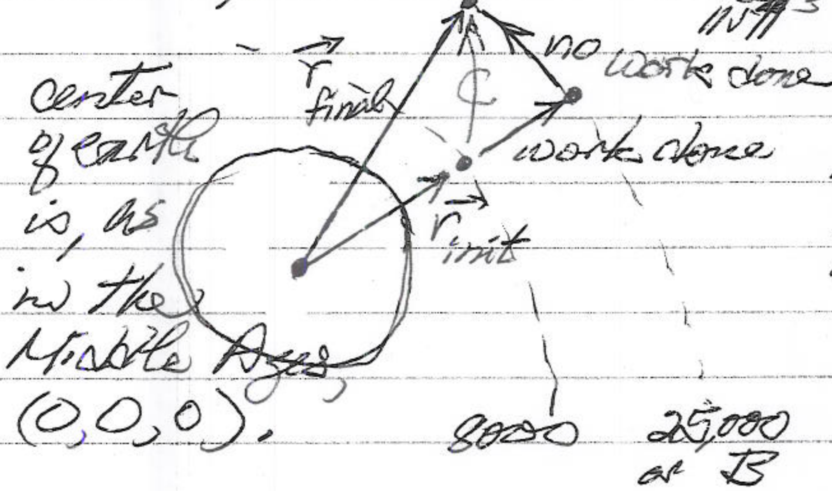
<u>Note</u>: To make sense an integral must have the form

$$\int h(\text{variable})\, d(\text{variable}).$$

Boy, did I ever mess up!!

7.2 #17 or WS4 #24, 25.

These results are still valid even if the particle doesn't move straight out: any crosswise motion it makes from one distance out to another will be $\perp \vec{r}$ and hence $\perp \vec{F}$, so no work is done. This is best seen by seeing that $\vec{F} = -\frac{GMm}{\|\vec{r}\|^3}\vec{r}$ is actually a gradient v.f. and we have independence of path. Our guess for the potential function was $f = GMm/\|\vec{r}\|$



and I tried to show that this works without using (x, y, z) coordinates to the extent possible. I did this by making up a "product rule" which was baloney — just flat out WRONG, as I saw the minute I got back to the office. Were you able to spot the error? No surprise then that it was not listed among the vector identities! OK, here goes:

depends on dist from $\vec{O}$.

$$f(x,y,z) = \frac{GMm}{\|\vec{r}\|} \quad \text{where } \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$$

$$\vec{\nabla} f = GMm\left(\frac{-1}{\|\vec{r}\|^2}\right)\vec{\nabla}(\|\vec{r}\|) \quad \text{by the Chain Rule}$$

I am using (x,y,z) but only in the easiest possible way.

Now $\vec{\nabla}(\|\vec{r}\|^2) = 2\|\vec{r}\|\,\vec{\nabla}(\|\vec{r}\|)$ "

But also $\vec{\nabla}(\|\vec{r}\|^2) = \vec{\nabla}(x^2+y^2+z^2)$
$= (2x, 2y, 2z) = 2(x,y,z) = 2\vec{r}$.

Putting these two computations together, we ~~get~~ $2\|\vec{r}\|\,\vec{\nabla}(\|\vec{r}\|) = 2\vec{r}$, or

$\vec{\nabla}(\|\vec{r}\|) = \frac{1}{\|\vec{r}\|}\vec{r}$ = unit vector in direction of $\vec{r}$. (This makes sense geometrically: the magnitude of the gradient is the maximum directional derivative, in the direction of the greatest rate of change. The greatest rate of change of $\|\vec{r}\|$ obviously takes place if you go in the direction of $\vec{r}$, and in this direction $\frac{d}{d\|\vec{r}\|}(\|\vec{r}\|)$

$= 1 = \left\|\frac{1}{\|\vec{r}\|}\vec{r}\right\| = \|\vec{\nabla}(\|\vec{r}\|)\|$.)

Back to our problem: so far

$$\vec{\nabla} f = \frac{-GMm}{\|\vec{r}\|^2}\vec{\nabla}(\|\vec{r}\|)$$

$$= -\frac{GMm}{\|\vec{r}\|^2}\left(\frac{1}{\|\vec{r}\|}\vec{r}\right) \quad \text{by the above,}$$

$$= -\frac{GMm}{\|\vec{r}\|^3}\vec{r} = \vec{F} \quad \text{as given}$$