Quiz 6 – Solution

1. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$

(a) Find a fundamental matrix.

$$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4).$$

$\lambda_1 = 2$: $(A - 2I)\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

$\lambda_2 = 4$: $(A - 4I)\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$$\Psi(t) = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}.$$

(b) Find $\Phi(t)$:

$$\Psi(0) = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \qquad \Psi(0)^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}.$$

$$\Phi(t) = \Psi(t)\,\Psi(0)^{-1} = \begin{pmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -1/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \\ -\frac{3}{2}e^{2t} + \frac{3}{2}e^{4t} & \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \end{pmatrix}.$$