# Exam 3 - Solution Key

1. $9y''+12y'+4y=0$ , $y(0)=a>0$, $y'(0)=-1$.

(a) $y=e^{rt}$; $9r^2+12r+4=0$

$(3r+2)^2=0 \qquad r=-\frac{2}{3}, -\frac{2}{3}$.

$y=c_1 e^{-\frac{2}{3}t}+c_2 t e^{-\frac{2}{3}t}$ $\qquad y' = -\frac{2}{3}c_1 e^{-\frac{2}{3}t}+c_2 e^{-\frac{2}{3}t}-\frac{2}{3}c_2 t e^{-\frac{2}{3}t}$

To satisfy $y(0)=a$ : $c_1=a$

$y'(0)=-1$ : $-\frac{2}{3}c_1+c_2=-1 \Rightarrow c_2=-1+\frac{2}{3}c_1=-1+\frac{2}{3}a$.

$\therefore y(t)= a e^{-\frac{2}{3}t}+\left(\frac{2}{3}a-1\right) t e^{-\frac{2}{3}t}$.

(b) The dominant term (as $t\to+\infty$) is the one with $te^{-\frac{2}{3}t}$

When that coefficient is negative, solutions will $\to -\infty$ (which requires that the solution becomes ~~positive~~ negative).

When the coefficient is positive, everything in both terms is positive so the sum is positive (always).

The coefficient of $te^{-2/3 t}$ is $\frac{2}{3}a-1$.

$\frac{2}{3}a-1=0 \iff \frac{2}{3}a=1 \iff a=\frac{3}{2}$.

2. $A=\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

(a) E-values & e-vectors: $\det(A-\lambda I)=\det\begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix}=(2-\lambda)(-2-\lambda)+3$
$=\lambda^2-1=(\lambda+1)(\lambda-1)=0$

$\lambda_1=1$: $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\vec v=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore \vec v^{(1)}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=-1$: $\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\vec v=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore \vec v^{(2)}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\Psi(t)=\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$. $\qquad \Psi(0)=\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

(b) $\Phi(t)=\Psi(t)\Psi(0)^{-1}=$ ~~$\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 3e^t & -e^{-t} \\ -e^t & e^t \end{bmatrix}$~~ $=\begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$

~~$=\frac{1}{2}\begin{bmatrix} 3 & -1 \end{bmatrix}$~~ $=\frac{1}{2}\begin{bmatrix} 3e^t-e^{-t} & -e^t+e^{-t} \\ 3e^t-3e^{-t} & -e^t+3e^{-t} \end{bmatrix}$.

(c) $\vec{x}(t) = \Phi(t)\begin{bmatrix}2\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3e^{t}-e^{-t} & -e^{t}+e^{-t}\\ 3e^{t}-3e^{-t} & -e^{t}+3e^{-t}\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix}6e^{t}-2e^{-t}+e^{t}-e^{-t}\\ 6e^{t}-6e^{-t}+e^{t}-3e^{-t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}7e^{t}-3e^{-t}\\ 7e^{t}-9e^{-t}\end{bmatrix}$$

$$= \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{t} - \frac{3}{2}\begin{bmatrix}1\\3\end{bmatrix}e^{-t}$$

3. We are told that, in effect, one solution is $\vec{x}^{(1)} = \begin{bmatrix}2\\1\end{bmatrix}e^{-t}$. Since there is not a second (independent) eigenvector, the second solution will be $\vec{x}^{(2)}(t) = \vec{v}^{(1)}te^{-t} + \vec{u}e^{-t}$ where $\vec{u}$ is any solution to $(A-\lambda_1 I)\vec{u} = \vec{v}^{(1)}$. That is

$\begin{bmatrix}-1/2 & 1\\ -1/4 & 1/2\end{bmatrix}\vec{u} = \begin{bmatrix}2\\1\end{bmatrix}$ $\quad -\frac{1}{2}u_1 + u_2 = 2. \Rightarrow u_2 = 2+\frac{1}{2}u_1 \Rightarrow \vec{u} = \begin{bmatrix}u_1\\ 2+\frac{1}{2}u_1\end{bmatrix}$

With $u_1 = 0$ this becomes $\vec{u} = \begin{bmatrix}0\\2\end{bmatrix}$, and so

$$\vec{x}^{(2)}(t) = \begin{bmatrix}2\\1\end{bmatrix}te^{-t} + \begin{bmatrix}0\\2\end{bmatrix}e^{-t} = \begin{bmatrix}2t\\ t+2\end{bmatrix}e^{-t}$$

4. The general solution to the homogeneous equation is: $\vec{x}_h = \Psi(t)\vec{c}$. A particular solution is $\vec{x}_p = \Psi(t)\vec{u}(t)$ where $\Psi(t)\vec{u}'(t) = \begin{bmatrix}e^{2t}\\ e^{-t}\end{bmatrix}$.

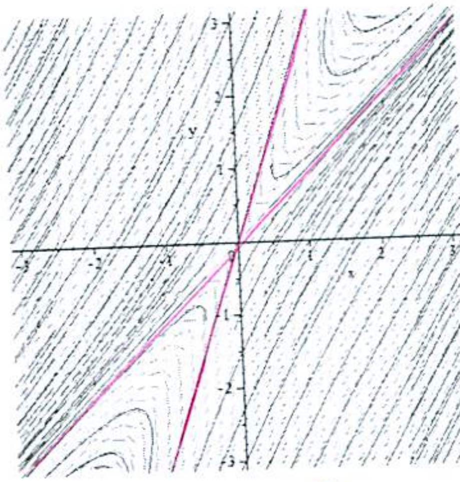
$$\vec{u}'(t) = \Psi(t)^{-1}\begin{bmatrix}e^{2t}\\ e^{-t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3e^{-t} & -e^{-t}\\ -e^{t} & e^{t}\end{bmatrix}\begin{bmatrix}e^{2t}\\ e^{t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3e^{t}-1\\ -e^{3t}+e^{2t}\end{bmatrix}$$

$u_1' = \frac{1}{2}(3e^{t}-1) \Longrightarrow u_1 = \frac{3}{2}e^{t} - \frac{t}{2}.$

$u_2' = \frac{1}{2}(-e^{3t}+e^{2t}) \Longrightarrow u_2 = -\frac{1}{6}e^{3t} + \frac{1}{4}e^{2t}$

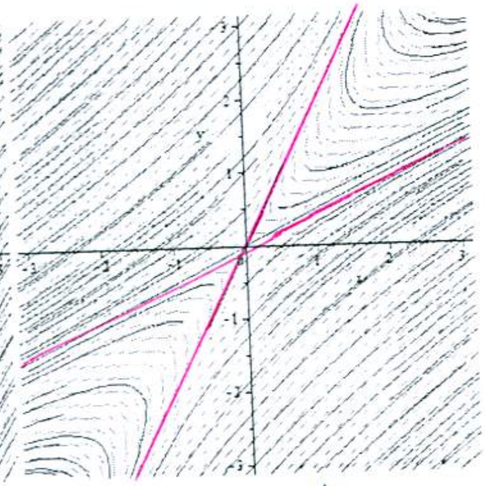
so $\vec{x}_p(t) = \Psi(t)\vec{u}(t) = \begin{bmatrix}e^{t} & e^{-t}\\ e^{t} & 3e^{-t}\end{bmatrix}\begin{bmatrix}\frac{3}{2}e^{t} - \frac{t}{2}\\ -\frac{1}{6}e^{3t}+\frac{1}{4}e^{2t}\end{bmatrix}$

$$= \begin{bmatrix}\frac{3}{2}e^{2t} - \frac{1}{2}te^{t} - \frac{1}{6}e^{2t} + \frac{1}{4}e^{t}\\ \frac{3}{2}e^{2t} - \frac{1}{2}te^{t} - \frac{1}{2}e^{2t} + \frac{3}{4}e^{t}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}e^{2t} + \left(\frac{1}{4}-\frac{t}{2}\right)e^{t}\\ e^{2t} + \left(\frac{3}{4}-\frac{t}{2}\right)e^{t}\end{bmatrix}$$



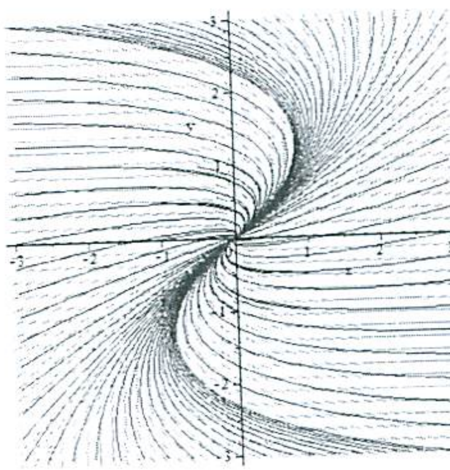
( i) System: a
Type: saddle
Stability: unstable

* straight-line solutions are along $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$



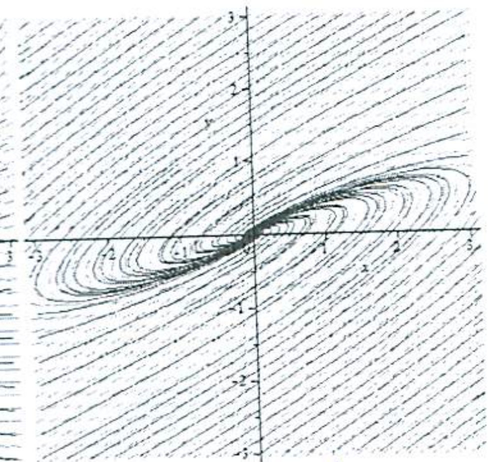
( ii) System: b
Type: saddle
Stability: unstable

* straight line solutions are along $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$



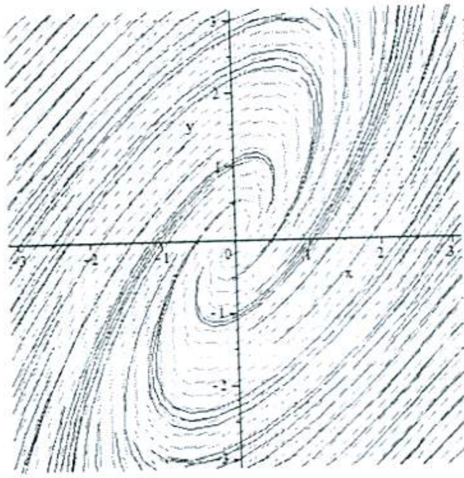
(iii) System: c
Type: improper node
Stability: asymptotically stable

* notice the heavy line along the vector $\begin{bmatrix}2\\1\end{bmatrix}$ (an e-vector) and at the origin.



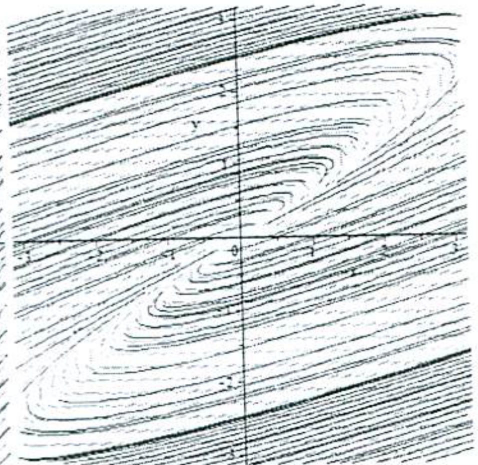
( iv) System: d
Type: spiral
Stability: asymptotically stable

* notice how there is a pile-up at the origin & that there appears to be a "blob" at the eq. solution.



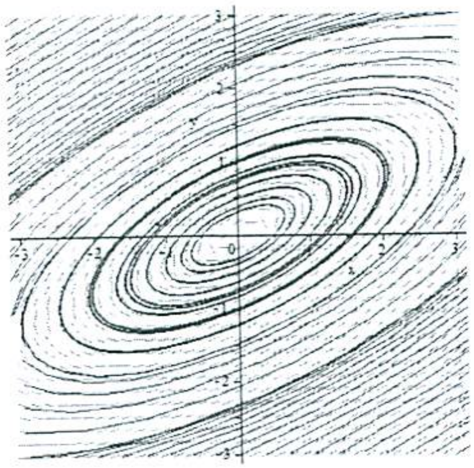
( v) System: e
Type: spiral
Stability: unstable

* solutions are moving away from the origin



( vi) System: f
Type: improper node
Stability: unstable

* solutions moving away from the origin & not a spiral



(vii) System: g
Type: center
Stability: stable

* purely ~~complex~~ imaginary eigenvalues.