MATH 520 **Spring, 2012** **Exam #1** **Name:** M^2

There are 100 points. **For full credit you must show your work.**

1. (15 points) Solve the equation $y' - (2/t)y = t\ln t$ with initial condition $y(1) = 4$.

Linear first order $\mu = e^{\int -\frac{2}{t}dt}$

$= e^{-2\ln t} = e^{\ln t^{-2}} = \frac{1}{t^2}$ integrating factor

$\frac{1}{t^2}y = \mu y = \int (t\ln t)(\frac{1}{t^2})\,dt$

$= \int \frac{\ln t}{t}\,dt = \frac{1}{2}(\ln t)^2 + C$

$y = t^2(\frac{1}{2}(\ln t)^2 + C)$ $\quad 4 = y(1) = 1(0 + C)$

$y = t^2(\frac{1}{2}(\ln t)^2 + 4)$

2. (15 points) Solve the equation $y' = ty^2/(3+t^2)$ with initial condition $y(1) = -1/\ln 6$.

Separate vars: $\frac{dy}{dt} = \frac{ty^2}{3+t^2}$ $\quad \int \frac{dy}{y^2} = \int \frac{t}{3+t^2}\,dt$

$-\frac{1}{y} = \frac{1}{2}\ln(3+t^2) + C$

$+\ln 6 = \frac{1}{2}\ln(4) + C$ $\quad +\ln 6 = \ln 2 + C$

$\ln 6 - \ln 2 = \ln 3 = C$ $\quad y = \dfrac{-1}{\frac{1}{2}\ln(3+t^2) + \ln 3}$

3. (15 points) Find the general solution of $y'' - y' - 2y = 8e^{3t}$.

Find y_H: $r^2 - r - 2 = 0$ $\quad (r-2)(r+1) = 0$

$y_1 = e^{2t}$; $y_2 = e^{-t}$ $\quad r=2 \quad r=-1$

Find y_p of form Ae^{3t}: $\quad y_p = Ae^{3t}$

$y_p'' - y_p' - 2y_p = (9A - 3A - 2A)e^{3t}$ $\quad y_p' = 3Ae^{3t}$

$= 8e^{3t}$ $\quad y_p'' = 9Ae^{3t}$

$4A = 8 \quad A = 2 \quad y = c_1y_1 + c_2y_2 + y_p = c_1e^{2t} + c_2e^{-t} + 2e^{3t}$

4. (20 points) a. Solve the equation $y'' + 2y' + 10y = 0$ with $y(0) = 6$ and $y'(0) = 0$.

   b. If this equation instead had a right hand side of $e^t cos(2t)$ what test formula would you use for the particular solution if the method of undetermined coefficients were to be used (but do NOT carry it out)?

$$r^2 + 2r + 10 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 40}}{2} = (-2 \pm \sqrt{-36})/2 = -1 \pm 3i$$

Fundamental solutions $y_1 = e^{-t}\cos 3t;\ y_2 = e^{-t}\sin 3t$

$$y = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t = 6 \text{ at } t=0$$

$$c_1 = 6$$

used $\begin{cases}\cos 0 = 1\\ e^0 = 1\\ \sin 0 = 0\end{cases}$

$$y' = c_1((e^{-t})(-3\sin 3t) + -e^{-t}\cos 3t)$$

$$+(c_2)(3e^{-t}\cos 3t + -e^{-t}\sin 3t) = 0 \text{ at } t=0$$

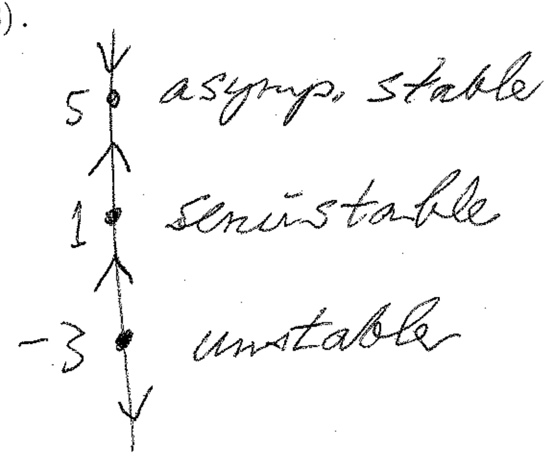
$$-c_1(0+1) + c_2(3+0) = 0$$

$$c_2 = 2$$

$$y = 6e^{-t}\cos 3t + 2e^{-t}\sin 3t$$

$$y_p = Ae^t\cos 2t + Be^t\sin 2t$$

5. (10 points) Construct the phase line and classify the equilibrium values as asymptotically stable, semi-stable, or unstable for the equation $y' = -(y-5)(y-1)^2(y+3)$.



6. (25 points) Solve $y'' - 2y' + y = e^t/(1+t^2)$, assuming that $y_1 = e^t$ and $y_2 = te^t$ are two given solutions of the corresponding homogeneous equation. First verify that these are indeed a fundamental set of solutions. Use the method of variation of parameters.

$$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = te^{2t} + e^{2t} - te^{2t} = e^{2t} \neq 0$$

$$y_1 = u_1 y_1 + u_2 y_2 \text{ with } \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \frac{e^t}{1+t^2} \end{cases}$$

$$\begin{cases} u_1' e^t + u_2' te^t = 0 \\ u_1' e^t + u_2'(te^t + e^t) = \frac{e^t}{1+t^2} \end{cases}$$ Divide through by e^t

$$(-)\begin{cases} u_1' + u_2' t = 0 \\ u_1' + u_2'(t+1) = \frac{1}{1+t^2} \end{cases}$$

$$u_2' t + u_2' - u_2' t = \frac{1}{1+t^2}$$

$$u_2' = \frac{1}{1+t^2} \qquad u_2 = \text{Arctan } t$$

$$u_1' = -u_2' t = -\frac{t}{1+t^2}$$

$$u_1 = -\frac{1}{2}\ln(1+t^2)$$

$$y = -\frac{1}{2}\ln(1+t^2)e^t + (\text{Arctan } t)te^t + c_1 e^t + c_2 te^t = y_p + y_H$$