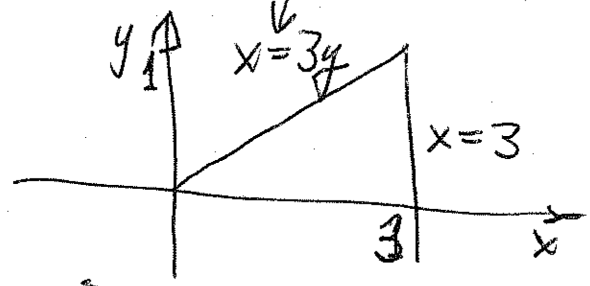
MATH 241 **Spring, 2010** **Exam #3** **Name:** M^2

For full credit you must show sufficient work that the method of obtaining your answer is clear. There are 100 points. Some reference formulas: $r = \rho \sin\phi$, $z = \rho\cos\phi$, $x = r\cos\theta$, $y = r\sin\theta$.

1. (15 points) Re-express the integral $\int_0^1 \int_{3y}^3 \cos(x^2)\,dx\,dy$ with the order of integration reversed, and compute the easier version.

or $y = \frac{1}{3}x$



$$\int_0^3 \int_0^{x/3} \cos(x^2)\,dy\,dx$$

$$= \int_0^3 y\cos(x^2)\Big|_{y=0}^{y=\frac{x}{3}} dx = \int_0^3 \frac{x}{3}\cos(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

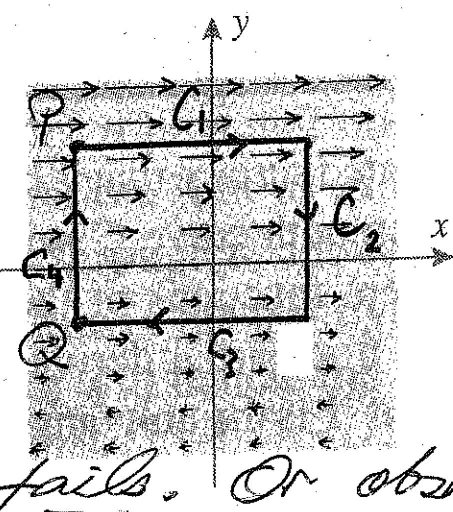
$$= \frac{1}{3}\left(\frac{1}{2}\right)\int_0^9 \cos u\,du$$

$$= \frac{1}{6}\sin u\Big|_0^9 = \frac{1}{6}\sin 9$$

so $\int_P^Q \vec{F}\cdot d\vec{r}$ does depend on the path.

2. (10 points) A vector field **F** is illustrated below, with a path C made up of C_1, C_2, C_3 and C_4 in this order. Determine if $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is negative, positive, or zero. Does **F** have the "independence of path" property in the region shown? Explain, briefly.

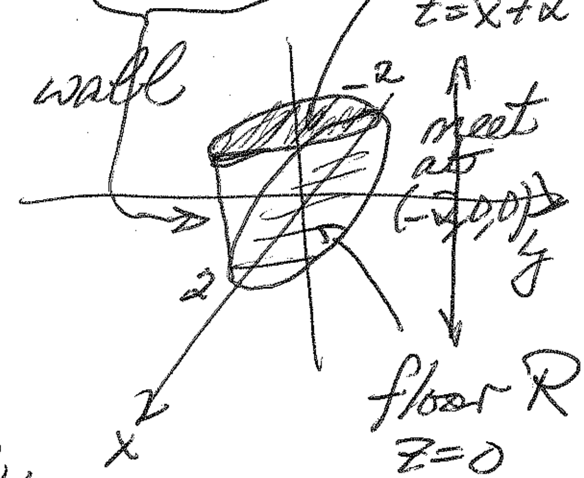
$$\oint_C \vec{F}\cdot d\vec{r} > 0$$



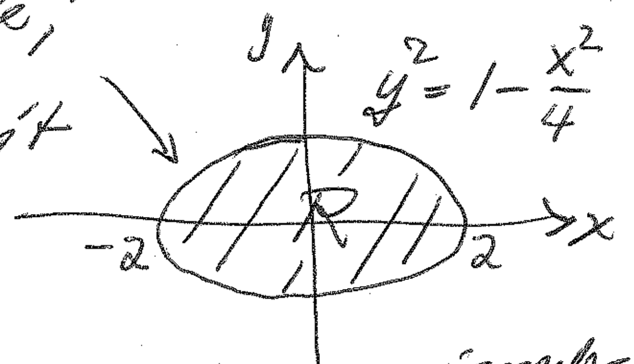
Since the integral around the loop is $\neq 0$, $\vec{F}$ is not conservative, and independence of path fails. Or observe that $\int_{C_1+C_2+C_3} \vec{F}\cdot d\vec{r} > 0$ but $\int_{-C_4} \vec{F}\cdot d\vec{r} = 0$

3. (15 points) SET UP completely (*i.e.*, give the limits of integration determined by D and an explicit form of dV, with everything in a consistent order); but do NOT compute, a triple integral for the volume of the region D bounded by the xy-plane, the plane $z = x + 2$, and inside the cylinder $\underbrace{x^2 + 4y^2 = 4}_{\text{wall}}$.

$$V = \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \int_{0}^{x+2} dz\, dy\, dx$$



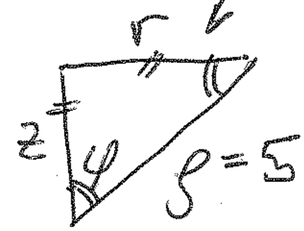
ellipse, not circle, so polar coord's don't work

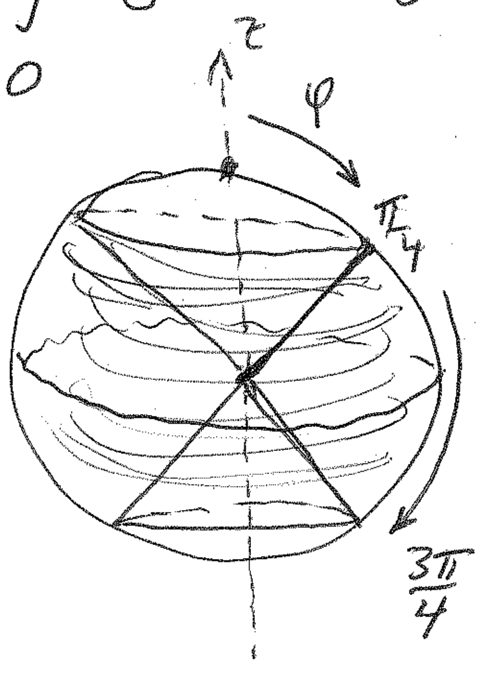


4. (10 points) SET UP, but do NOT compute, the triple integral in spherical coordinates that gives the volume inside the sphere $x^2 + y^2 + z^2 = 25$, and **outside** the double cone $z = \pm r$ (*i.e.*, $z^2 = x^2 + y^2$).

$$V = \int_{0}^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{5} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

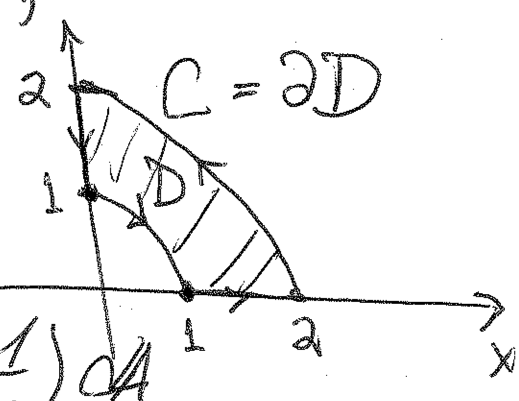
isosceles rt. Δ





If $z = r$, $\cos\varphi = \sin\varphi$ and $\varphi = \frac{\pi}{4}$. Symmetrically below $\varphi = \frac{3\pi}{4}$.

5. (25 points) Compute $\oint_C \mathbf{G} \cdot d\mathbf{r}$ for $\mathbf{G} = \langle 3xy^2, -3x^2y \rangle$, where C is given by the circular arc of $x^2 + y^2 = 1$ clockwise from $(0,1)$ to $(1,0)$, then the segment from $(1,0)$ to $(2,0)$, then the circular arc of $x^2 + y^2 = 4$ counterclockwise from $(2,0)$ to $(0,2)$, and finally the segment from $(0,2)$ back to $(0,1)$. (Hint: use a major result and then use polar coordinates.)



$$M = 3xy^2 \qquad N = -3x^2y$$

$$\frac{\partial M}{\partial y} = 6xy \qquad \frac{\partial N}{\partial x} = -6xy$$

Green's Thm

$$\oint_C \vec{G} \cdot d\vec{r} \overset{\downarrow}{=} \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \iint_D -12xy \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \overbrace{-12(r\cos\theta)(r\sin\theta)}^{-12r^3\cos\theta\sin\theta} \, r \, dr \, d\theta$$

$$= -12 \int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4} \Big|_1^2 \right) \sin\theta \cos\theta \, d\theta$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= -3 \, (16-1) \, \frac{\sin^2\theta}{2} \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{45}{2}$$

$M \quad N$

6. (25 points) Let $\mathbf{F} = \langle ye^{xy}, xe^{xy} \rangle$ and C be the path given by by $x = 1 - 2t$, $y = -1 - t$ for $0 \le t \le 1$.

   a. What is the domain of $\mathbf{F}$? At what point P does C begin and at what point Q does it end? $\mathbb{R}^2$ $\quad P(1,-1) \quad Q(-1,-2)$

   b. Is the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path? Briefly explain.

product rule!

$\mathbb{R}^2$ is simply conn.

$$\frac{\partial N}{\partial x} = x(e^{xy}y) + e^{xy}$$
$$\frac{\partial M}{\partial y} = y(e^{xy}x) + e^{xy}$$

equal so $\vec{F}$ is conservative and indep of path holds.

   c. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. This can be done by straightforward direct calculation, or if you can justify that there is a potential function $\varphi(x, y)$ for $\mathbf{F}$, then compute it and use it.

$\vec{F}$ is conservative, so $\vec{F} = \vec{\nabla}\varphi$ for some $\varphi(x,y)$.

$$\varphi(x,y) = \int y e^{xy}\,dx = e^{xy} + A(y)$$
$$\varphi(x,y) = \int x e^{xy}\,dy = e^{xy} + B(x)$$

So $\varphi(x,y) = e^{xy} + C$

(Check $\frac{\partial \varphi}{\partial x} = e^{xy}y = M$, $\frac{\partial \varphi}{\partial y} = e^{xy}x = N$)

Then
$$\int_C \vec{F}\cdot d\vec{r} = \int_P^Q \vec{\nabla}\varphi \cdot d\vec{r}$$
$$= \varphi(Q) - \varphi(P)$$
$$= e^{2} - e^{-1} = e^{2} - \frac{1}{e}$$

6. (25 points) Let $\mathbf{F} = \langle ye^{xy}, xe^{xy} \rangle$ and C be the path given by by $x = 1 - 2t$, $y = -1 - t$ for $0 \le t \le 1$.

a. What is the domain of $\mathbf{F}$? At what point P does C begin and at what point Q does it end?

b. Is the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path? Briefly explain.

$$dx = -2dt$$
$$dy = -dt$$

c. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. This can be done by straightforward direct calculation, or if you can justify that there is a potential function $\varphi(x, y)$ for $\mathbf{F}$, then compute it and use it.

Alternatively $\int \vec{F} \cdot d\vec{r} = \int_C y e^{xy} dx + x e^{xy} dy$

$$= \int_0^1 (-1-t)e^{(-1-t)(1-2t)}(-2dt) + (1-2t)e^{(-1-t)(1-2t)}(-dt)$$

$$= \int_0^1 (2+2t-1+2t)e^{-1-t+2t+2t^2}\,dt$$

$$= \int_0^1 e^{2t^2+t-1}(4t+1)\,dt$$

$u = 2t^2 + t - 1$
$du = (4t+1)dt$

$$= \int_{-1}^{2} e^u\,du$$

$$= e^2 - e^{-1} = e^2 - \frac{1}{e}$$