MATH 172 Fall, 2009 Quiz #6 Name: M^2

For full credit you must show sufficient work to justify your answer. You may use the following: (1) an affine continuous model $\frac{dQ}{dt} = aQ + b$ has an explicit solution $Q(t) = Ce^{at} + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition; (2) an affine discrete model $Q_{n+1} = aQ_n + b$ has an explicit solution $Q_n = Ca^n + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition.

1. a. Formulate an affine discrete model in which 80% of drug in the bloodstream from one day to the next is used up, but the remainder is reinforced by a maintenance dose of 40 mg/day. Let A_n denote the amount of drug present in the bloodstream on day n.

20% is left for next day

$$\underbrace{A_{n+1}}_{\text{new amount}} = \underbrace{0.2 A_n}_{\text{amount left}} + \underbrace{40}_{\text{maintenance dose}}$$

b. What is the equilibrium (steady state) amount of the drug in the bloodstream?

$A^* = A_n = A_{n+1}$

$$A^* = 0.2A^* + 40$$
$$0.8A^* = 40$$
$$A^* = 50$$

c. Give an explicit formula for A_n if the initial dose is $A_0 = 10$ mg.

$A_n = C(0.2)^n + 50$ general form of solution

Put $n=0$: $10 = A_0 = C(1) + 50 \Rightarrow C = -40$

$$\boxed{A_n = 50 - 40(0.2)^n}$$

c. Compute A_1, A_2, A_{10}, A_{20} either from the formula of part (c) or from the model of part (a) with the help of your calculator.

$A_1 = 42$

$A_2 = 48.4$

$A_{10} = 50$

$A_{20} = 50$

If instead $A_0 = 60$, then $C = +10$ and $A_n = 50 + 10(0.2)^n$. In this case $A_1 = 52$, $A_2 = 50.4$, $A_{10} = 50$, $A_{20} = 50$

(over →)

d. Describe the long term behavior of A_n : does it increase, decrease, oscillate, tend towards or away from the equilibrium? Does the equilibrium value you found in part (b) appear to be stable or unstable? Explain verbally and / or graphically.

Also note $(0.2)^n \to 0$ as $n \to \infty$, so less & less is subtracted from 50.

A_n increases steadily, rising up to equil. of $A^* = 50$. This equil appears to be stable; at least if we start below it we rise back to it. (Bonus if you also discuss what happens if $A_0 > 50$.)

2. In a good, but shrinking, wooded habitat, a population of snakes $S = S(t)$ growing at a **per capita rate** of 3% yr $^{-1}$, but 54 migrate away over the course of each year. Write an affine continuous model equation for this situation, and solve it, assuming that the initial snake population is 1,600. What exactly happens to the snake population in the long term, and how do you know? Bonus: if the population is growing, compute the doubling time; if the population is shrinking, compute the extinction time.

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$$\frac{dS}{dt} = \underbrace{0.03S}_{\text{growth rate}} - \underbrace{54}_{\text{loss rate}}$$ all in snakes/yr

(Note: $\frac{dS}{dt} < 0$ at $t=0$ when $S=1600$)

S^* found at $\frac{dS}{dt} = 0 = 0.03S - 54$

$$S^* = \frac{54}{0.03} = 1800$$

$$S(t) = C e^{0.03t} + 1800$$ general form of solution

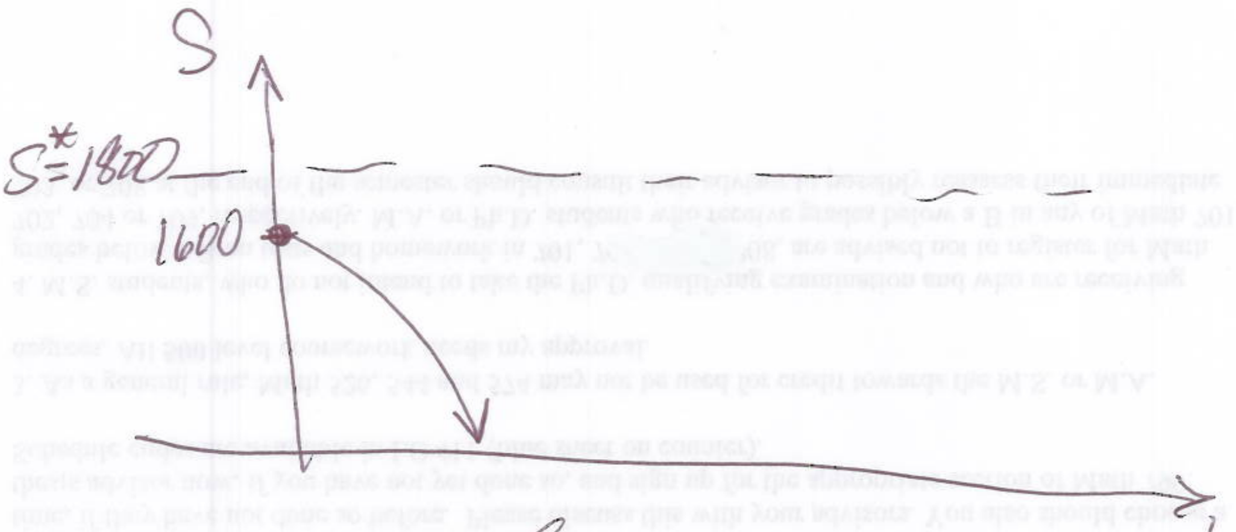
$$1600 = S(0) = Ce^0 + 1800 = C + 1800$$

$$C = -200$$

$$\boxed{S(t) = -200\underbrace{e^{0.03t}}_{\to \infty \text{ as } t \to \infty} + 1800}$$

Since we are subtracting more and more from 1800, eventually $S(t) = 0$ (extinction)

Also note $\frac{dS}{dt} < 0$ (so $S(t)$ is ↓) when $S < 1800$.



$S(t)=0$ when

$$0=-200e^{0.03t}+1800$$

$$200e^{0.03t}=1800\,,\quad e^{0.03t}=9$$

$$0.03t=\ln(9)$$

$$t=\frac{\ln(9)}{0.03}=73.24$$

Q26. More on #1a.

Another way to think about this model is

$$\Delta A_n = \text{gain} - \text{loss}$$
$$= 40 - 0.8A_n$$

or

$$A_{n+1} - A_n = 40 - 0.8A_n$$

$$\boxed{\begin{array}{c} A_{n+1} = A_n - 0.8A_n + 40 \\ \text{new amount} = \text{old amount} - \text{loss} + \text{gain} \end{array}}$$

This simplifies to what we had before

$$\boxed{A_{n+1} = 0.2A_n + 40}$$

$$\boxed{\text{new amount} = \text{remaining amount} + \text{gain}}$$

Note: If you had (incorrectly) $A_{n+1} = 0.8A_n + 40$ you should have found $0.2A^* = 40$, or $A^* = 200$.

If you had $A_{n+1} = -0.8A_n + 40$ you should have had $1.8A^* = 40$ and $A^* = 22.2$. I gave credit if you followed through.