MATH 172 Fall, 2007 Exam #3 Name: M^2

There are 100 points. **For full credit you must show your work.** Recall that an affine discrete dynamic model $Q(t+1) = aQ(t) + b$ has an explicit solution $Q(t) = Ca^t + Q^*$, where Q^* is the equilibrium value, and C can be determined from the initial condition.

1. (15 points) We are given a discrete model $P(n+1) = (-0.9)P(n) + 95$ with $P(0) = 70$.

   a. Find the explicit solution for $P(n)$.

$P^* = -0.9P^* + 95$

$1.9P^* = 95$

$P^* = 50$

$C = 20$

$\boxed{P(n) = 20(-0.9)^n + 50}$

$P(n) = C(-0.9)^n + 50$

$70 = P(0) = C + 50$

   b. What happens to $P(n)$ as $n \to \infty$? Does it increase, decrease, oscillate, tend towards or away from the equilibrium? Conclude whether the equilibrium is stable or not.

$P(n)$ oscillates with smaller and smaller oscillations around $P^* = 50$. As $n \to \infty$, $P(n) \to 50$. We conclude the equil. is stable, as this* is true no matter what C is.

* $P(n) \to 50$ as $n \to \infty$

2. (15 points) Let $A = \begin{bmatrix} -4 & 6 \\ 9 & 11 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Determine which one of $\mathbf{v}$ and $\mathbf{w}$ is an eigenvector for A, and find the corresponding eigenvalue.

$A\vec{v} = \begin{bmatrix} 2 \\ 20 \end{bmatrix} \neq$ multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (slope 10 vs. 1)

$A\vec{w} = \begin{bmatrix} 14 \\ -7 \end{bmatrix} = -7\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -7\vec{w}$

$\vec{w}$ is an eigenvector with eigenvalue -7.

3. (7 points) Compute the sum of the series $\sum_{n=0}^{\infty}(5/2)(-2/3)^n$ or state that no sum exists; and explain why or why not.

$$\frac{5}{2} + \frac{5}{2}\left(-\frac{2}{3}\right) + \frac{5}{2}\left(-\frac{2}{3}\right)^2 + \cdots \qquad a = \frac{5}{2} \quad r = -\frac{2}{3}$$

$|r| < 1$ so the sum is $\frac{a}{1-r} = \frac{5/2}{1+2/3} = \frac{5/2}{5/3} = \boxed{\frac{3}{2}}$

4. (14 points) Compute the equilibrium point (E, F) of the dynamical system

$$u_n = 3u_{n-1} - 2v_{n-1} - 4$$
$$v_n = 5u_{n-1} - 3v_{n-1} - 28$$

$\boxed{(20, 18)}$

$$\begin{cases} u^* = 3u^* - 2v^* - 4 \\ v^* = 5u^* - 3v^* - 28 \end{cases}$$

$$-2\begin{cases} 4 = 2u^* - 2v^* \\ 28 = 5u^* - 4v^* \end{cases}$$

$2v^* = 2u^* - 4$

$v^* = u^* - 2$

$= 18$

$$\begin{cases} -8 = -4u^* + 4v^* \\ 28 = 5u^* - 4v^* \end{cases}$$

$20 = u^*$

As an alternative method of solution you could put this in place of v^* in the 2nd eqn; then solve for u^*.

5. (12 points) A matrix M has eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. These go with eigenvalues $\lambda_1 = 1.04$ and $\lambda_2 = 0.8$, respectively. We have $\mathbf{P}_0 = \mathbf{v}_1 + 5\mathbf{v}_2$.

a. Compute $\mathbf{P}_1 = M\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

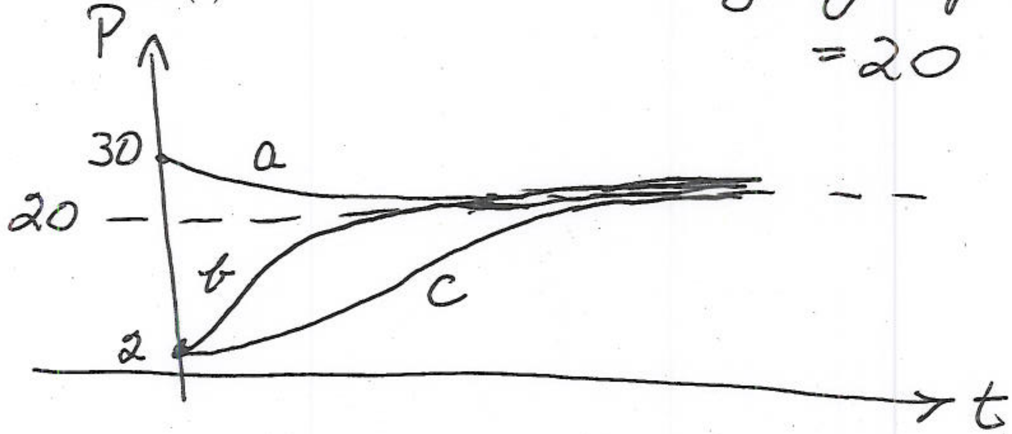
$$\vec{P}_1 = M\vec{P}_0 = M(\vec{v}_1 + 5\vec{v}_2) = M\vec{v}_1 + 5M\vec{v}_2$$
$$= \boxed{(1.04)\vec{v}_1 + 5(0.8)\vec{v}_2} = 1.04\vec{v}_1 + 4\vec{v}_2$$

b. Compute $\mathbf{P}_2 = M^2\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

$$M^2\vec{P}_0 = M\vec{P}_1 = \boxed{\begin{aligned}&(1.04)^2\vec{v}_1 + 5(0.8)^2\vec{v}_2 \\ =\ &1.08\vec{v}_1 + 3.2\vec{v}_2\end{aligned}}$$

6. (12 points) Sketch, on the same graph, P as a function of t if $\frac{dP}{dt} = rP(1-\frac{P}{20})$. Label your graphs with a, b, c.
   a. if $r = 0.6$ and $P(0) = 30$
   b. if $r = 0.6$ and $P(0) = 2$
   c. if $r = 0.3$ and $P(0) = 2$

logistic model
carrying capacity
$= 20$



7. (25 points) A population consists of 0 to 3 year olds: newborns (N_t), juveniles (J_t), subadults (S_t), and adults (A_t).
   a. In each time period an individual either dies or survives and moves into the next age group. Set up the Leslie or population transition matrix A to express the following data. Newborns have a mortality rate of 80% over the first year. Juveniles have a survival rate of 60%, and subadults have a survival rate of 90%. Subadults produce 1 newborn on average each year; adults produce 10, and die after reproduction.

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \end{bmatrix}$$

   b. The initial population vector is $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 0 \\ 10 \\ 0 \end{bmatrix}$. Compute $\mathbf{P}_1$ and $\mathbf{P}_2$.

$$\vec{P}_1 = A\vec{P}_0 = \begin{bmatrix} 10 \\ 20 \\ 0 \\ 9 \end{bmatrix} \qquad \vec{P}_2 = A\vec{P}_1 = \begin{bmatrix} 90 \\ 2 \\ 12 \\ 0 \end{bmatrix}$$

c. Now suppose these groups represent stages of development. Modify the transition matrix A to give a Lefkowitch matrix B in which the fecundities (reproduction of newborns) are as before, 25% of juveniles survive as juveniles, and 35% grow into subadults. Subadults survive as subadults with a 75% chance and grow into adults with a 15% chance. Adults survive with a 90% chance.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 0.2 & .25 & 0 & 0 \\ 0 & .35 & .75 & 0 \\ 0 & 0 & .15 & .9 \end{bmatrix}$$

d. The dominant eigenvalue for A is $\lambda = 1.048$, which goes with an eigenvector $\mathbf{v} = \begin{bmatrix} 386 \\ 74 \\ 42 \\ 36 \end{bmatrix}$. We find that $\mathbf{P}_{30} = \begin{bmatrix} 221.5 \\ 30.3 \\ 27.2 \\ 15.7 \end{bmatrix}$ (total 294.7). Has the population reached its stable age distribution? Explain.

SAD comes from normalizing $\vec{v}$

Total = 538. $\begin{bmatrix} 0.717 \\ 0.138 \\ 0.078 \\ 0.067 \end{bmatrix}$ At $t=30$ distrib. is $\begin{bmatrix} 0.752 \\ 0.103 \\ 0.092 \\ 0.053 \end{bmatrix}$

These are clearly unequal, so P_{30} is not at SAD.

e. The dominant eigenvalue for B is 1.172 and the total population at $t = 30$ is 5117. Why, in real life terms, should these values be larger than the corresponding ones for A?

Organisms live longer and hence reproduce more over their lifetimes, so the growth rate is higher, and the population grows to higher levels in scenario B. For example a mature adult produces $10 + (0.9)(10) + (0.9)^2(10) + \cdots = \frac{10}{1-0.9}$ $= 100$ offspring over her expected lifetime.

8. (8 Bonus points) In a predator-prey continuous model system, the prey ("rabbits") population $R(t)$, measured in hundreds, grows with a per capita rate of 0.16 yr $^{-1}$ in the absence of predators ("foxes"). The net growth rate is reduced by predation: each possible rabbit-fox interaction results on average in the loss of 0.8 rabbits (they don't all interact, and it isn't always fatal if they do). Model this with a mass action interaction with a coefficient of 0.8 and an appropriate plus or minus sign. The fox population $F(t)$, measured in tens, declines at a per capita rate of 0.25 yr $^{-1}$ in the absence of rabbits. The net growth rate, however, is increased by predation: each fox-rabbit interaction increases the fox population on average by 0.1 (in other words it takes consumption of 10 rabbits to produce one fox).

a. Write the model equations for this system.

$$\frac{dR}{dt} = 0.16R - 0.8RF$$

$$\frac{dF}{dt} = -0.25F + 0.1RF$$

b. One equilibrium is of course $R = 0 = F$. Find another one.

$$\begin{cases} 0 = 0.16R - 0.8RF \\ 0 = -0.25F + 0.1RF \end{cases}$$

$$0 = (0.16 - 0.8F)R$$

$$0 = (-0.25 + 0.1R)F$$

So $R = F = 0$ is one possibility.

The other is

$0.16 = 0.8F$ so $F = 0.2$ tens or 2

$0.25 = 0.1R$ so $R = 2.5$ hundreds or 250