MATH 172 Fall, 2011 Exam #2 Name: $M^2$

There are 100 points. **For full credit you must show your work.** Recall that (1) an affine continuous model $\frac{dQ}{dt} = aQ + b$ has an explicit solution $Q(t) = Ce^{at} + Q^*$, where $Q^*$ is the equilibrium value, and $C$ can be determined from the initial condition, and (2) an affine discrete model $Q_{n+1} = aQ_n + b$ has an explicit solution $Q_n = Ca^n + Q^*$, where $Q^*$ is the equilibrium value, and $C$ can be determined from the initial condition.

1. (16 points) We are given a discrete model $P_{n+1} = (-0.9)P_n + 95$ with $P(0) = 70$.

   a. Find the explicit solution for $P_n$.

   Find the equilibrium $P^* = -0.9P^* + 95$

   $1.9P^* = 95$

   $P^* = 50$

   Then by (2) above

   $P_n = C(-0.9)^n + 50$

   $70 = P_0 = C(-0.9)^0 + 50 = C + 50$

   $C = 20$

   $P_n = 20(-0.9)^n + 50$

   b. What happens to $P_n$ as $n \to \infty$? Does it increase, decrease, oscillate, tend towards or away from the equilibrium? Conclude whether the equilibrium is stable or not.

   $P_n \to 50$ as $n \to \infty$ by smaller and smaller oscillations. We conclude $P^* = 50$ is stable.

2. (10 points) Let $A = \begin{bmatrix} -4 & 6 \\ 9 & 11 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Determine which (possibly both) of $\mathbf{v}$ and $\mathbf{w}$ is an eigenvector for $A$, and find the corresponding eigenvalue.

   $A\vec{v} = \begin{bmatrix} 14 \\ 42 \end{bmatrix} = 14\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so $\vec{v}$ is an eigenvector with eigenvalue 14.

   $A\vec{w} = \begin{bmatrix} 14 \\ -7 \end{bmatrix} = -7\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ so $\vec{w}$ is an eigenvector with eigenvalue $-7$.

3. (16 points) In a particular wooded habitat, a population of birds $B = B(t)$ **declines** at a **per capita rate** of 6% yr $^{-1}$, but is reinforced by the in-migration of birds from destroyed nearby habitats at 300 birds/yr. Write a continuous affine model equation for this situation, and solve it, assuming that the initial bird population is 15,000. What exactly happens to the bird population in the long term, and how do you know?

$B' = -0.06B + 300$. Find equilibrium

$B' = 0$: $\quad B^* = \frac{300}{0.06} = 5000$

$B = Ce^{-0.06t} + 5000$

$15000 = Ce^{0} + 5{,}000 \qquad B = 10000e^{-0.06t} + 5000$

$C = 10{,}000$

Since $e^{-0.06t} \to 0$ as $t \to \infty$, $B(t) \to$ 5000

4. (16 points) A matrix $M$ has eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. These go with eigenvalues $\lambda_1 = 1.04$ and $\lambda_2 = 0.8$, respectively. We have $\mathbf{P}_0 = \mathbf{v}_1 + 5\mathbf{v}_2$.

a. Compute $\mathbf{P}_1 = M\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

$$\vec{P}_1 = M\vec{P}_0 = M(\vec{v}_1 + 5\vec{v}_2)$$
$$= M\vec{v}_1 + 5M\vec{v}_2$$
$$= 1.04\vec{v}_1 + 5(0.8)\vec{v}_2$$

b. Compute $\mathbf{P}_2 = M^2\mathbf{P}_0$. You may leave the symbols $\mathbf{v}_1$ and $\mathbf{v}_2$ in your answer.

$$\vec{P}_2 = M\vec{P}_1 = 1.04M\vec{v}_1 + 5(0.8)M\vec{v}_2$$
$$= (1.04)^2\vec{v}_1 + 5(0.8)^2\vec{v}_2$$

c. What is a good approximation to $\mathbf{P}_t = M^t\mathbf{P}_0$ when $t$ is large? Briefly explain.

$\vec{P}_t = (1.04)^t\vec{v}_1 + 5(0.8)^t\vec{v}_2$. Since $(0.8)^t \to 0$ as $t \to \infty$,

$\vec{P}_t \approx (1.04)^t\vec{v}_1$

5. (26 points) A population consists of 0 to 3 year olds: newborns ( $N_t$ ), juveniles ( $J_t$ ), subadults ( $S_t$ ), and mature adults ( $M_t$ ).

a. *In each time period an individual either dies or survives and moves into the next age group.* Set up the Leslie or population transition matrix $A$ to express the following data. Newborns have a mortality rate of 80% over the first year. Juveniles have a survival rate of 60%, and subadults have a survival rate of 90%. Subadults produce 1 newborn on average each year; mature adults produce 10, and die after reproduction.

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \end{bmatrix}$$

b. The initial population vector is $\mathbf{P}_0 = \begin{bmatrix} 100 \\ 0 \\ 10 \\ 0 \end{bmatrix}$. Compute $\mathbf{P}_1$ and $\mathbf{P}_2$.

$$\vec{P}_1 = A\vec{P}_0 = \begin{bmatrix} 10 \\ 20 \\ 0 \\ 9 \end{bmatrix} \qquad \vec{P}_2 = A\vec{P}_1 = \begin{bmatrix} 90 \\ 2 \\ 12 \\ 0 \end{bmatrix}$$

c. The dominant eigenvalue for $A$ is $\lambda = 1.048$, which goes with an eigenvector $\mathbf{v} = \begin{bmatrix} 0.717 \\ 0.138 \\ 0.078 \\ 0.067 \end{bmatrix}$. We find that $\mathbf{P}_{30} = \begin{bmatrix} 221.5 \\ 30.3 \\ 27.2 \\ 15.7 \end{bmatrix}$ (total 294.7).

Has the population reached its stable age distribution? Explain.

$$\vec{D}_{30} = \begin{bmatrix} 0.752 \\ 0.103 \\ 0.092 \\ 0.053 \end{bmatrix} \neq \vec{v}$$

Since $\vec{v}$ gives the SAD, we are not there yet with $\vec{P}_{30}$.

d. Now suppose these groups represent stages of development. Modify the matrix $A$ to give a Lefkowitch matrix $B$ in which the fecundities (reproduction of newborns) are as before, the survival of newborns to become juveniles remains the same, and 25% of juveniles survive as juveniles, while 35% grow into subadults. Subadults remain subadults with a 75% chance and grow into mature adults with a 15% chance. Mature adults survive with a 90% chance.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 0.2 & 0.25 & 0 & 0 \\ 0 & 0.35 & 0.75 & 0 \\ 0 & 0 & 0.15 & 0.9 \end{bmatrix}$$

e. The dominant eigenvalue for $B$ is 1.172 with eigenvector $\mathbf{v} = \begin{bmatrix} 0.668 \\ 0.145 \\ 0.120 \\ 0.066 \end{bmatrix}$,

$1.172 > 1.048$

and the total population at $t = 30$ is 5117 with $\mathbf{P}_{30} = \begin{bmatrix} 3420.3 \\ 741.9 \\ 615.3 \\ 339.3 \end{bmatrix}$. Has this

$5117 > 295$

population reached stable age distribution, and why? Use the eigenvalue to predict the total population at $t = 31$. Why, in real life terms, should the eigenvalue and the population values be larger than the corresponding ones for $A$?

$$\vec{D}_{30} = \begin{bmatrix} 0.668 \\ 0.145 \\ 0.120 \\ 0.066 \end{bmatrix} = \vec{v}$$

So pop has reached SAD

$$\text{Total}_{31} = (1.172)\,\text{Total}_{30} = 5997$$

Since reproductive stages last more than one time period, subadults and mature adults reproduce over and over again. This leads to a faster growth rate for the whole population (a bigger eigenvalue) and a bigger population.

6. (16 points) A grasshopper population $G = G(t)$, measured in millions, now finds that the quality of the habitat is decreasing as measured by the per capita growth rate $s(t)$; in fact $\frac{dG}{dt} = s(t)G$, with $s(t) = 0.08 - 0.005t$ in units of $\text{yr}^{-1}$. At time $t = 0$ there were 20 million individuals.

a. Use separation of variables to solve the model equation and use your solution to find the number of grasshoppers at times $t = 10$ years.

$$\int \frac{dG}{G} = \int (0.08 - 0.005t)\,dt$$

$$\ln G = 0.08t - 0.005\frac{t^2}{2} + C_1 \qquad e^{C_1} = C$$

$$G = Ce^{0.08t - 0.005t^2/2}$$

$$20 = G(0) = Ce^0 = C \qquad G = 20e^{0.08t - 0.005\frac{t^2}{2}}$$

$$G(10) = 20e^{(0.08)(10) - (0.005)(100)/2}$$

$$= 34.67 \text{ million}$$

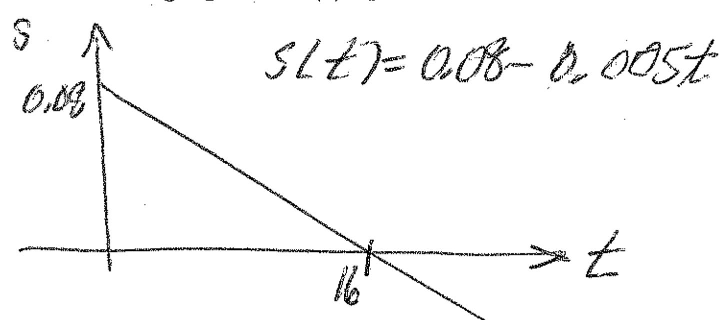
b. How does the information provided about $s(t)$ tell us that the quality of the habitat is decreasing?

$s(t)$ is a decreasing function of $t$.

c. (Bonus) Explain how you can tell that $G(t)$ increases and then decreases. At what time does the population peak? Hint: the graph of $s(t)$ gives it away.

$s(t) = 0$ at

$t = \frac{0.08}{0.005} = 16$ years




Before then $s(t) > 0$,

So $\frac{dG}{dt} = s(t)G > 0$ and $G(t)$ is increasing.

After $t = 16$, $\frac{dG}{dt} = s(t)G < 0$ so $G(t)$ is decreasing. So $G(t)$ has a maximum at $t = 16$. (Of course a graph of $G(t)$ is OK.)