HW 9 - Solutions

#19. Claim: Let $f, g : D \to \mathbb{R}$ be uniformly continuous.
Then $f+g$ is uniformly continuous.

Proof: Let $\varepsilon > 0$ be given.

By uniform continuity of f:

$$\exists\, \delta_1 > 0 \text{ s.t. } x, y \in D,\ |x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

By uniform continuity of g:

$$\exists\, \delta_2 > 0 \text{ s.t. } x, y \in D,\ |x-y| < \delta_2 \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{2}.$$

Define $\delta = \min\{\delta_1, \delta_2\}$.

For any $x, y \in D$ with $|x-y| < \delta$ then

$$\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |f(x) + g(x) - (f(y) + g(y))| \\
&= |f(x) - f(y) + g(x) - g(y)| \\
&\le |f(x) - f(y)| + |g(x) - g(y)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

because $|x-y| < \delta \le \delta_1$

because $|x-y| < \delta \le \delta_2$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f+g$ is uniformly continuous. ☒

To see that the product fg of uniformly continuous functions need not be uniformly continuous, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. This function is uniformly continuous, but $f \cdot f$ is not uniformly continuous.

$$|x^2 - y^2| = |(x+y)(x-y)| = |x+y|\,|x-y| = \underbrace{|x+y|}_{\text{not bounded!}}\,\delta \quad (\text{when } |x-y| < \delta).$$

More generally,

$$\begin{aligned}
|(fg)(x) - (fg)(y)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\
&\le |f(x) - f(y)|\,|g(x)| + |f(y)|\,|g(x) - g(y)|
\end{aligned}$$

not bounded

#20. Claim: Let $f: A \to B$ and $g: B \to C$ be uniformly continuous.
Then $g \circ f: A \to C$ is uniformly continuous.

Proof: Let $\varepsilon > 0$ be given.

Because g is uniformly continuous on B
$\exists \delta^* > 0$ s.t. $|u-v| < \delta^*$, $u,v \in B \Rightarrow |g(u) - g(v)| < \varepsilon$.

Because f is uniformly continuous on A
$\exists \delta > 0$ s.t. $|x-y| < \delta$, $x,y \in A \Rightarrow |f(x) - f(y)| < \delta^*$.

Let $x,y \in A$ with $|x-y| < \delta$. Then $f(x), f(y) \in B$ and $|f(x) - f(y)| < \delta^*$.

Thus

$$|(g \circ f)(x) - (g \circ f)(y)| = |g(f(x)) - g(f(y))|$$
$$= |g(u) - g(v)|$$
where $u = f(x)$, $v = f(y)$
so $u,v \in B$, $|u-v| < \delta^*$
$$< \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $g \circ f$ is uniformly continuous. $\boxed{\equiv}$

#22. Let $f: (2,7) \to \mathbb{R}$ be defined by $f(x) = x^3 - x + 1$.
To see that f is uniformly continuous on $(2,7)$,

let $\varepsilon > 0$ be given.

When $x,y \in (2,7)$ and $|f(x) - f(y)| = |(x^3 - x + 1) - (y^3 - y + 1)|$

$= |x^3 - y^3 - x + y| = |(x-y)(x^2+xy+y^2) - (x-y)| = |x^2+xy+y^2-1|\,|x-y|$

The largest possible value of $|x^2+xy+y^2-1|$ occurs when $x=y=7$;
it is $|7^2+7^2+7^2-1| = 146$. This suggests choosing $\delta = \frac{\varepsilon}{146}$.

Now, to complete the proof: when $x,y \in (2,7)$ with $|x-y| < \delta = \frac{\varepsilon}{146}$.

$$|f(x) - f(y)| = |x^2+xy+y^2-1|\,|x-y| \le 146\,|x-y| < 146\,\delta = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof that this f is uniformly continuous on $(2,7)$.