# Solutions - HW #8

(p. 101)

#13. Let $f: D \to \mathbb{R}$ be continuous at $x_0 \in D$.
Prove there is a $M > 0$ and a neighborhood $Q$ of $x_0$
such that $|f(x)| \leq M$ for all $x \in Q \cap D$.

Proof: Let $\varepsilon = 1$.
Since $f$ is continuous as $x_0$ $\exists \delta > 0$ s.t. $|x - x_0| < \delta$, $x \in D$
$\Rightarrow |f(x) - f(x_0)| < 1$.

Let $Q = \{x : |x - x_0| < \delta\} = (x_0 - \delta, x_0 + \delta)$.

For $x \in Q \cap D$: $|f(x) - f(x_0)| < 1$.

$$-1 < f(x) - f(x_0) < 1.$$

$$f(x_0) - 1 < f(x) < f(x_0) + 1$$

Thus, taking absolute values: $|f(x)| < \max\{|f(x_0)+1|, |f(x_0) - 1|\}$

Define $M = \max\{|f(x_0) + 1|, |f(x_0) - 1|\}$.

so that $|f(x)| < M \quad \forall\, x \in Q \cap D$. $\boxed{\Xi}$

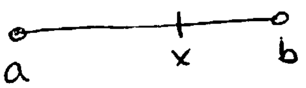
#15. Let $f, g: D \to \mathbb{R}$ be continuous on $D$.
The function $h: D \to \mathbb{R}$ defined by $h(x) = \max\{f(x), g(x)\}$
can also be written as $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$.
Now, because the sum of continuous functions is continuous, $\frac{1}{2}(f+g)$ is continuous on $D$.
Likewise $f - g$ is continuous and so is $\frac{1}{2}|f - g|$.
Lastly, the sum of $\frac{1}{2}(f+g)$ & $\frac{1}{2}|f-g|$ must be continuous because it's two terms are each ~~cont~~ continuous.

#16. For $x > 0$, $x^x = e^{x \ln x}$. For $x > 0$, $\ln$ is continuous and so
$x \to x \ln(x)$ is continuous for $x > 0$. Lastly, because exp
is continuous on all of $\mathbb{R}$, we conclude that $h(x)$ must be
a continuous solution on $(0, \infty)$.

#18. This piecewise-defined function was encountered in Chapter 2, Exercise 17 (p. 80). At that time we determined the function had a limit everywhere except the odd integers, where the function jumps from 1 on the left to −1 on the right. This jump prevents the function from being continuous everywhere except the negative integres

#27. Claim 1: Let $a, b \in \mathbb{R}$. The set $\{x : a < x < b\}$ is an open set.

Proof: To show $\{x : a<x<b\}$ is open we show that for every ~~$x$~~ with $a<x<b$ there is a neighborhood of $x$ that is also a subset of $\{x : a<x<b\}$.




Let $x$ be an element of $\{x : a<x<b\}$.
Define $\delta = \min\{\frac{1}{2}(b-x), \frac{1}{2}(x-a)\}$.
(Note that $\delta > 0$)
Define $Q = (x-\delta, x+\delta)$.

because $x > a$

Then $x - \delta > x - \frac{1}{2}(x-a) = \frac{1}{2}(x+a) > \frac{1}{2}(a+a) = a$
and $x+\delta < x + \frac{1}{2}(b-x) = \frac{1}{2}(b+x) < \frac{1}{2}(b+b) = b$

because $x < b$.

so $Q = (x-\delta, x+\delta) \subset (a,b) = \{x : a<x<b\}$.
Since $Q$ is an interval, it's a neighborhood of $x_0$. that is contained within $\{x : a<x<b\}$.
This holds for every $x \in \{x : a<x<b\}$ so this set is open. ☒

Claim 2: Let $a, b \in \mathbb{R}$. The set $\{x : a \le x \le b\}$ is closed.

Proof: Let $x$ be an element of $\{x : a \le x \le b\}$.
If $x \ne b$ then the sequence $\{x + \frac{1}{n}\}_{n=n_0}^{\infty}$ is in $\{x : a \le x \le b\}$ when $n_0$ is chosen so that $x + \frac{1}{n_0} \le b$.
This sequence converges to $x$, so $x$ is an accumulation pt.
This works for all $x$ except $x = b$. For this value, use the sequence $\{b - \frac{1}{n}\}_{n=n_0}^{\infty}$. These are in the set $\{x : a \le x \le b\}$ and converge to $b$, so $b$ is an accumulation pt of the set. Thus every member of $\{x : a \le x \le b\}$ is an accumulation pt of itself. Hence closed

#34. Let $G_n = (1+\frac{1}{n}, 2)$. for $n \geq 1$.
Then $\bigcup_{n=1}^{\infty} G_n = (1,2)$ so $\{G_n\}_{n=1}^{\infty}$ is an open cover of $(1,2)$. But no finite sub cover can cover $(1,2)$.