Solutions - HW#7 - Corrected (#11)

p. 104

#2. $f(x) = \begin{cases} \dfrac{2x^2-18}{x+3} & x \neq -3 \\ -12 & x = -3 \end{cases}$.

To see that f is continuous at -3 we will show that $\lim_{x\to -3} f(x)$ exists and equals $f(-3)$.

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{2x^2-18}{x+3} = \lim_{x\to -3} \frac{2(x+3)(x-3)}{x+3} \overset{x \neq -3}{=} \lim_{x\to -3} \frac{2(x-3)}{1} = 2(-3-3) = -12 = f(-3).$$

#5. $f(x) = \frac{1}{\sqrt{x}} - \frac{\sqrt{x+1}}{\sqrt{x}}$ for $x \in (0,1)$.

If we can make f continuous at 0, then $f(0)$ must be $\lim_{x\to 0} f(x)$.

Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\sqrt{x+1}}{\sqrt{x}} \cdot \frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} = \lim_{x\to 0} \frac{1-(x+1)}{\sqrt{x}(1+\sqrt{x+1})} = \lim_{x\to 0} \frac{-x}{\sqrt{x}(1+\sqrt{x+1})}$

$$= \lim_{x\to 0} \frac{-\sqrt{x}}{1+\sqrt{x+1}} = \frac{0}{2} = 0.$$

Since this limit exists, defining $f(0) = 0$ will make the function f continuous on $[0,1)$.

#7. $f(x) = \begin{cases} x^2 & x \text{ rational} \\ ? & x \text{ irrational} \end{cases}$ is a continuous function (on $\mathbb{R}$).

Knowing that f is continuous at $x = \sqrt{2}$ we know that for any a sequence $\{x_n\}$ with $x_n \to \sqrt{2}$ it is also true that $f(x_n) \to f(\sqrt{2})$.

Now, $\sqrt{2} = 1.414$ (as an irrational number, this is an infinite nonrepeating decimal)

Define $\{x_n\}_{n=1}^{\infty}$ so that $x_1 = 1.4$, $x_2 = 1.41$, $x_3 = 1.414$,

That is, x_n is the decimal approximation to $\sqrt{2}$ correct to n decimal places.

Then $\lim_{n\to\infty} x_n = \sqrt{2}$. Computing: $f(x_1) = 1.4^2 = 1.96$

$f(x_2) = 1.41^2 = 1.9881$

$f(x_3) = 1.414^2 = 1.999396$

It is clear that $\lim_{n\to\infty} f(x_n) = (\sqrt{2})^2 = 2$, so $f(\sqrt{2}) = 2$.

#8. Let f be a continuous function on (a,b) with $f(x)=0$ for all rational numbers in (a,b).

Claim: $f(x)=0$ for all $x \in (a,b)$.

Proof: Let $x \in (a,b)$ be given.

If x is rational, then $f(x)=0$ by definition (of f).

If x is irrational, then (much like in #7), there is a sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$ that converges to x. Since $f(x_n)=0$ for all $n=1,2,\ldots$, $\lim_{n\to\infty} f(x_n)=0$.

So, to have f continuous at x we must have $f(x)=0$.

Since every $x \in (a,b)$ is either rational or irrational, we are done. $\boxed{\Xi}$

#9. $f(x) = x\sin\left(\frac{1}{x}\right)$ for $x \in (0,1)$.

While $\lim_{x\to 0} \frac{1}{x}$ and $\lim_{x\to 0} \sin\frac{1}{x}$ do not exist, $\sin\left(\frac{1}{x}\right)$ is bounded,

$$-1 \le \sin\left(\tfrac{1}{x}\right) \le 1$$

So $\quad -x \le x\sin\left(\frac{1}{x}\right) \le x.$

Now, because $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$, we see that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

So, defining $f(0)=0$ makes f continuous at 0.

#11. $f(x) = \begin{cases} 8x & \text{if } x \text{ is rational} \\ 2x^2+8 & \text{if } x \text{ is irrational.} \end{cases}$

Note that $\lim_{x\to 2} 8x = 16 = \lim_{x\to 2} 2x^2+8$ and $\lim_{x\to 1} 8x = 8$ while $\lim_{x\to 1} 2x^2+8 = 10$

For $x_0=2$. To prove f is continuous at $x=2$, let $\varepsilon>0$ be given.

Let $\delta = \min\left\{\frac{\varepsilon}{8}, \frac{\varepsilon}{10}\right\} = \frac{\varepsilon}{10}$.

For $|x-2|<\delta$, x is either rational or irrational.

If x is rational and $|x-2|<\delta$, then $|f(x)-f(2)| = |8x-16| = 8|x-2|$

$< 8\delta = 8\cdot\frac{\varepsilon}{10} = \frac{4}{5}\varepsilon < \varepsilon$.

If x is irrational and $|x-2|<\delta$ then $|f(x)-f(2)| = |2x^2+8-16|$

$= |2x^2-8| = 2|x+2||x-2| < 2(5)\delta = 10\delta = 10\cdot\frac{\varepsilon}{10} = \varepsilon$.

(Notice that I bound $|x+2|$ by 5 for all x near 2.)

Since $|f(x)-f(2)| < \varepsilon$ for any x (rational or irrational)

f is continuous at $x=2$.

To show f is not continuous at 1 because, with $\varepsilon = 1/2$

there is no $\delta > 0$ for which both $|x-1| < \delta$ & $|f(x)-f(1)| < \varepsilon$.

Notice that $f(1) = 8$ because $x=1$ is rational.

When x is irrational,

$$|f(x)-f(1)| = |2x^2 + 8 - 8| = 2|x|^2.$$

and, when x is rational

$$|f(x)-f(1)| = |8x-8| = 8|x-1| = 8\delta < \varepsilon \text{ if } \delta < \frac{\varepsilon}{8}.$$

So, only the irrational points are a problem.

To finish, suppose $\delta^* = \min\{\delta, \frac{1}{2}\}$.

Then
$$\begin{aligned} |x-1| < \delta^* &\iff -\delta^* < x-1 < \delta^* \\ &\iff \underbrace{1-\delta^*}_{> \frac{1}{2}} < x < \underbrace{1+\delta^*}_{< 2} \end{aligned}$$

so $\frac{1}{2} < x < 2$.

and $\frac{1}{4} < |x|^2 < 4$.

Back to the irrational case,

$$|f(x)-f(1)| = 2|x|^2 \geq 2(1-\delta^*)^2 > 2\left(\frac{1}{2}\right)^2 = 1/2 = \varepsilon.$$

so these values do <u>not</u> converge to $f(1) = 8$ at $x=1$.