Solutions HW#6

p.79

#10. Consider $x_n = 2^{-n} = \frac{1}{2^n}$ for $n = 1, 2, \ldots$

Then $x_n \to 0$ as $n \to \infty$.

$$f(x_n) = (2^{-n})^{2^{-n}} = 2^{-n2^{-n}}$$

Notice that $\lim\limits_{n\to\infty} n2^{-n} = \lim\limits_{n\to\infty} \frac{n}{2^n} = 0$.

Thus $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} 2^{-n2^{-n}} = 2^0 = 1$.

#17. Let $f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is } \cancel{\text{odd}} \text{ even} \\ x - [x+1] & \text{if } [x] \text{ is } \cancel{\text{even}} \text{ odd.} \end{cases}$

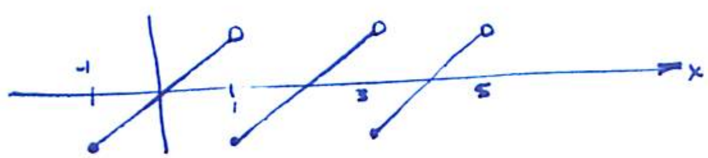
For $0 \le x < 1$, $[x] = 0$ is even so $f(x) = x - 0 = x$

For $1 \le x < 2$, $[x] = 1$ is odd so $f(x) = x - 2 = x-2$

For $2 \le x < 3$, $[x] = 2$ is even so $f(x) = x - 2 = x - 2$

For $3 \le x < 4$, $[x] = 3$ is odd so $f(x) = x - 4 = x - 4$

In general, $f(x) = \begin{cases} \vdots & \\ x & \text{if } -1 \le x < 1 \\ x-2 & \text{if } 1 \le x < 3 \\ x-4 & \text{if } 3 \le x < 5 \\ \vdots & \end{cases}$



From the graph we see that f has a limit at every point except the odd integers.

#19. Let $f(x) = \frac{\sqrt{9-x} - 3}{x}$ for $x \in (0,1)$.

Notice that $f(x) = \frac{\sqrt{9-x} - 3}{x} \cdot \frac{(\sqrt{9-x} + 3)}{(\sqrt{9-x} + 3)} = \frac{-x}{x(\sqrt{9-x}+3)} = \frac{-1}{\sqrt{9-x}+3}$

So, by the properties of limits (because $\lim\limits_{x\to 0} (\sqrt{9-x} + 3) = 6 \ne 0$),

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{-1}{\sqrt{9-x}+3} = \frac{\cancel{\lim\limits_{x\to 0}} -1}{\lim\limits_{x\to 0}(\sqrt{9-x}+3)} = \frac{-1}{6}.$$

#22. Let $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$ and $g(x) = -f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x > 0 \end{cases}$.

Then $(f+g)(x) = f(x) + g(x) = 0$. for all x

Here $\lim_{x \to 0} f(x)$ & $\lim_{x \to 0} g(x)$ do not exist but $\lim_{x \to 0} (f+g)(x) = 0$.

With these same functions,

$$(fg)(x) = f(x)\,g(x) = -1 \quad \text{for all } x$$

so $\lim_{x \to 0} (fg)(x) = -1$.

Similarly, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = -1$ for all x.

so $\lim_{x \to 0} \left(\frac{f}{g}\right)(x) = -1$.