Solutions - HW #5

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#2. $f : (-2,0) \to \mathbb{R}$ $\quad f(x) = \dfrac{2x^2+3x-2}{x+2}$

To determine if f has a limit at $x=-2$, note that (for $x \neq -2$):

$$f(x) = \frac{2x^2+3x-2}{x+2} = \frac{(x+2)(2x-1)}{x+2} = 2x-1 \longrightarrow \underset{L}{\overset{\|}{-5}} \text{ as } x \to -2.$$

To prepare for the proof, suppose $0<|x+2|<\delta$. Then

$$|f(x)-L| = \left|\frac{2x^2+3x-2}{x+2}+5\right| = |(2x-1)+5| = |2x+4| = 2|x+4| < 2\delta$$

This suggest choose $2\delta = \varepsilon$, i.e. $\delta = \frac{\varepsilon}{2}$.

Proof: Let $\varepsilon > 0$ be given.

Define $\delta = \varepsilon/2$.

For $0<|x+2|<\delta$: $\displaystyle |f(x)-L| = \left|\frac{2x^2+3x-2}{x+2}+5\right|$

$$= |(2x-1)+5|$$

$$= |2x+4|$$

$$= 2|x+2| < 2\delta = \varepsilon$$

Because $\varepsilon>0$ is arbitrary, we conclude $\lim\limits_{x\to -2} f(x) = -5$. ☐

#4. $f(x) = \begin{cases} 1 & x \geq -2 \\ -1 & x < -2 \end{cases}$ is defined for all $x \in \mathbb{R}$, is bounded by $+1 / -1$.

For $x_0 > -2$: $\lim\limits_{x\to x_0} f(x) = 1$

For $x_0 < -2$: $\lim\limits_{x\to x_0} f(x) = -1$.

But, if $x=-2$. I claim the limit does not exist.

Suppose $\lim\limits_{x\to -2} f(x) = L$.

With $\varepsilon = \frac{1}{2}$, $\exists \delta > 0$ s.t. $0<|x+2|<\delta \Rightarrow |f(x)-L| < \varepsilon = \frac{1}{2}$.

Let $p = -2 + \delta/2$ & $q = -2 - \delta/2$.

Then $f(p) = 1$ and $f(q) = -1$ but also $|f(p)-L| < \frac{1}{2}$ & $|f(q)-L| < \frac{1}{2}$.

$2 = |f(p)-f(q)| = |f(p)-L+L-f(q)| \leq |f(p)-L| + |f(q)-L| < \frac{1}{2}+\frac{1}{2} = 1$.

But it's not true that $2<1$, so we have a contradiction.

& so $\lim\limits_{x\to -2} f(x)$ does not exist.

#5. Let $f: D \to \mathbb{R}$ with x_0 an accumulation point of f.

Suppose $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} f(x) = L_2$ with $L_2 > L_1$.

Let $\varepsilon = \frac{1}{2}(L_2 - L_1)$. Then $\varepsilon > 0$.

By def. of limit: $\exists \delta_1 > 0$ s.t. $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$

and $\exists \delta_2 > 0$ s.t. $0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$

To have both of these true at the same time, let $\delta = \min\{\delta_1, \delta_2\}$.

Then
$$\begin{aligned}(L_2 - L_1) = |L_2 - L_1| &= |L_2 - f(x) + f(x) - L_1| \\ &\le |f(x) - L_2| + |f(x) - L_1| \\ \text{strict inequality} \rightarrow &< \varepsilon + \varepsilon \\ &= 2\varepsilon = L_2 - L_1\end{aligned}$$

But it's not true that $L_2 - L_1 < L_2 - L_1$ so we have a contradiction.

$\therefore L_2 = L_1$. $\blacksquare$

#7. $f: (0,1) \to \mathbb{R}$, $f(x) = x \cos\left(\frac{1}{x}\right)$

Because $\lim_{x \to 0} x = 0$ and $\underbrace{-1 \le \cos\left(\frac{1}{x}\right) \le 1}_{\cos(\frac{1}{x}) \text{ is bounded}}$ for all $0 < x < 1$,

$\lim_{x \to 0} f(x) = 0$.

Justify: Suppose $0 < |x - 0| < \delta$, then $|f(x) - 0| = \left|x \cos\frac{1}{x}\right| = |x| \left|\cos\frac{1}{x}\right| \le |x| = \delta = \varepsilon$

(in the proof, choose $\delta = \varepsilon$).

#9. $f: (-1,1) \to \mathbb{R}$, $f(x) = \frac{x+1}{x^2-1}$.

Since $x \ne \pm 1$: $f(x) = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$ which is not bounded for x near 1.

Because f is unbounded near $x = 1$, $\lim_{x \to 1} f(x)$ does not exist.