Solutions - HW #4

p. 55 #16.

Claim: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are Cauchy, so is $\{a_nb_n\}_{n=1}^{\infty}$.

Thinking: Our goal is to use $|a_n - a_m| < \varepsilon' \ \forall n,m \geq N_1$

& $|b_n - b_m| < \varepsilon'' \ \forall n,m \geq N_2$

to show that $|a_nb_n - a_mb_m| < \varepsilon \ \forall n,m \geq N$.

Now, $|a_nb_n - a_mb_m| = |a_nb_n - a_nb_m + a_nb_m - a_mb_m|$

$\leq |a_nb_n - a_nb_m| + |a_nb_m - a_mb_m|$

$= |a_n|\,|b_n - b_m| + |a_n - a_m|\,|b_m|$

Assume $|a_n| \leq M'$, $|b_n| \leq M''$

$< M'\varepsilon'' + M''\varepsilon'$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ if $\varepsilon'' = \frac{\varepsilon}{2M'}$ & $\varepsilon' = \frac{\varepsilon}{2M''}$.

Note: You can also assume $\varepsilon'' = \varepsilon'$.

Then it will be most natural to use $\varepsilon' = \varepsilon'' = \frac{\varepsilon}{M' + M''}$.

In either case, you would need $m,n \geq N = \max\{N_1, N_2\}$.

Proof: Because $\{a_n\}_{n=1}^{\infty}$ is Cauchy, it is bounded, say $|a_n| \leq M' \ \forall n \in J$.

Likewise, $\{b_n\}_{n=1}^{\infty}$ is bounded, so $|b_n|_{n=1}^{\infty} \leq M'' \ \forall n \in J$.

Let $\varepsilon > 0$ be given.

Because $\{a_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_1 \in J$ s.t. $|a_n - a_m| < \frac{\varepsilon}{M' + M''} \ \forall n,m \geq N_1$

and because $\{b_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_2 \in J$ s.t. $|b_n - b_m| < \frac{\varepsilon}{M' + M''} \ \forall n,m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. For all $m,n \geq N$:

$|a_nb_n - a_mb_m| = |a_nb_n - a_nb_m + a_nb_m - a_mb_m|$

$\leq |a_n|\,|b_n - b_m| + |a_n - a_m|\,|b_m| < M'\frac{\varepsilon}{M' + M''} + M''\frac{\varepsilon}{M' + M''} = \varepsilon.$

Because $\varepsilon>0$ is arbitrary, we have shown that
$$\forall \varepsilon>0\ \exists N\in\mathbb{J} \text{ s.t. } |a_nb_n-a_mb_m|<\varepsilon.$$
$\therefore\ \{a_nb_n\}_{n=1}^{\infty}$ is a Cauchy sequence. $\boxed{\text{LE}}$.

#18. The simplest example is probably $\{(-1)^n\}_{n=1}^{\infty}$ but many others exist.

#19. The example in #21 is a set with countably many accumulation points: $\{2^n+\frac{1}{k} : k,n\in\mathbb{J}\}$. (Accum. pts are $\{2^n : n\in\mathbb{J}\}$.)

#26. Any example that has cancellation between the two sequences: Let $\{a_n\}_{n=1}^{\infty}=\{n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}=\{-n+\frac{3}{2}\}_{n=1}^{\infty}$; then $\{a_n+b_n\}_{n=1}^{\infty}=\{\frac{3}{2}\}_{n=1}^{\infty}$.

#34. The even terms are the subsequence
$$\{a_{2k}\}_{k=1}^{\infty}=\left\{(-1)^{2k}\left(1-\frac{1}{2k}\right)\right\}_{k=1}^{\infty}=\left\{1-\frac{1}{2k}\right\}_{k=1}^{\infty}$$
This converges to 1.
Similarly, the sequence of odd terms is
$$\{a_{2k-1}\}_{k=1}^{\infty}=\left\{(-1)^{2k-1}\left(1-\frac{1}{2k-1}\right)\right\}_{k=1}^{\infty}=\left\{-\left(1-\frac{1}{2k-1}\right)\right\}_{k=1}^{\infty}.$$
(which converges to -1).

#40. Claim: $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing.
Proof: By induction: $P(n)$: $a_{n+1}\le a_n$: $P(1)$: $a_2=\sqrt{6+a_1}=\sqrt{12}<6=a_1$.
Assume $P(n)$ is true: $a_{n+1}\le a_n$. Then $P(n+1)$:
$$a_{n+2}=\sqrt{6+a_{n+1}}\le\sqrt{6+a_n}=a_{n+1}.$$
By induction, $P(n)$ is true for all $n\in\mathbb{J}$. $\square$

Claim: $\{a_n\}_{n\geq 1}^{\infty}$ is bounded.

Proof: Because this sequence is decreasing, an upper bound is $a_1 = 6$. A lower bound is 0 because $\sqrt{n} \geq 0$ $\forall n \in J$.

Therefore $0 \leq a_n \leq 6$. $\forall n \in J$. $\square$

Now, because $\{a_n\}_{n\geq 1}^{\infty}$ is bounded and monotone, it converges.

Let the limit be L.

Then the subsequence $\{a_{n+1}\}_{n\geq 1}^{\infty}$ also converges to L.

Also, $\{\sqrt{6+a_n}\}_{n\geq 1}^{\infty}$ converges to $\sqrt{6+L}$.

Then, since $a_{n+1} = \sqrt{6+a_n}$ $(\forall n \in J)$, $L = \sqrt{6+L}$.

$$L^2 = 6+L$$
$$L^2 - L - 6 = 0$$
$$(L-3)(L+2) = 0 \implies L = 3 \text{ or } L = -2.$$

only $L = 3$ is within the bounds for this sequence so the limit must be 3.

#36. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

Claim: $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence

Proof: There are 2 cases to consider, based on the number of values taken on by the sequence.
Let $S = \{a_n : n \in J\}$.

Case 1: S is an infinite set.
In this case S is a bounded infinite set of real numbers. By the Bolzano-Weierstrass Theorem S has an accumulation point, call it x_0.

Then by Theorem 1.17 $\exists$ sequence $\{x_n\}_{n=1}^{\infty}$ of numbers of S (other than x_0) that converges to x_0.
Since each of these x_n is the value of a_m for some $m \in J$, we see that this sequence is really a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Case 2: S is a finite set.
If S is finite, then there must be at least one of the values that occurs an infinite number of times in $\{a_n\}_{n=1}^{\infty}$. Choose the subsequence to be all the terms of the sequence that have this one value; this is a convergent subsequence of $\{a_n\}_{n=1}^{\infty}$.

Because S has either a finite or infinite number of elements, these 2 cases cover all possibilities. ☒