Solutions - HW3

§1.2 #14. Claim: Every Cauchy sequence is bounded.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence.

Let $\varepsilon = 1$. Because $\{a_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N \in J$ s.t. $|a_n - a_m| < 1$ for all $m, n \geq N$.

In particular, when $m = N$: $|a_n - a_N| < 1$ for all $n \geq N$.

Because $|a_n - a_N| < 1$: $-1 < a_n - a_N < 1$

$a_N - 1 < a_n < a_N + 1$. for all $n \geq N$.

Let $M = \max\{a_1, a_2, \ldots, a_{N-1}, a_N + 1\}$

and $m = \min\{a_1, a_2, \ldots, a_{N-1}, a_N + 1\}$

Then $m \leq a_n \leq M$ for all $n = 1, 2, \ldots, \ldots$

and so $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. ∎

#15. Claim: if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are Cauchy, then so is $\{a_n + b_n\}_{n=1}^{\infty}$.

Thinking ~~Proof~~: The definition of Cauchy says that

$\forall \varepsilon' > 0 \ \exists N_1 \in J$ s.t. $|a_n - a_m| < \varepsilon' \ \forall m, n \geq N_1$

$\forall \varepsilon' > 0 \ \exists N_2 \in J$ s.t. $|b_n - b_m| < \varepsilon' \ \forall m, n \geq N_2$.

We want to determine ε' & N. so that

$|(a_n + b_n) - (a_m + b_m)| < \varepsilon$ for all $m, n \geq N$.

Now,
$$\begin{aligned} |(a_n + b_n) - (a_m + b_m)| &= |(a_n - a_m) + (b_n - b_m)| \\ &\leq |a_n - a_m| + |b_n - b_m| \quad m,n \geq N_1 \ \& \ m,n \geq N_2. \\ &< \varepsilon' + \varepsilon' = 2\varepsilon' = \varepsilon \ \text{ if } \varepsilon' = \tfrac{\varepsilon}{2}. \end{aligned}$$

This suggests choosing $\varepsilon' = \frac{\varepsilon}{2}$ and $N = \max\{N_1, N_2\}$.

Proof: Let $\varepsilon > 0$ be given.

Because $\{a_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_1 \in J$ s.t. $|a_n - a_m| < \frac{\varepsilon}{2} \ \forall m, n \geq N_1$

& because $\{b_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N_2 \in J$ s.t. $|b_n - b_m| < \frac{\varepsilon}{2} \ \forall m, n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. For any $m, n \geq N$ we know that $m, n \geq N_1$ & $m, n \geq N_2$ so that

$$|(a_n + b_n) - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Because $\varepsilon > 0$ ~~was chosen~~ is arbitrary, this works for all $\varepsilon > 0$ and so $\{a_n + b_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ∎

#17. Claim: $\left\{\frac{2n+1}{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence.

Thinking: We have to show that $\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon \quad \forall m,n \geq N$.

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|2+\frac{1}{n} - (2+\frac{1}{m})\right| = \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

$$\leq \varepsilon \iff \frac{2}{N} = \varepsilon \iff N = \frac{2}{\varepsilon}.$$

Proof: Let $\varepsilon > 0$ be given.

Let $N = \frac{2}{\varepsilon}$ (well, the 1st positive integer larger than $\frac{2}{\varepsilon}$).

For $m,n \geq N$:

$$\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| = \left|2+\frac{1}{n} - \left(2+\frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{1}{n} + \left(\frac{-1}{m}\right)\right|$$

$$\leq \left|\frac{1}{n}\right| + \left|\frac{-1}{m}\right| = \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

$$< \frac{2}{2/\varepsilon} = \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, for any $\varepsilon > 0$ there is an $N \in J$ s.t. $\left|\frac{2n+1}{n} - \frac{2m+1}{m}\right| < \varepsilon$ for all $m,n \geq N$, i.e. $\left\{\frac{2n+1}{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. $\boxed{\varepsilon}$

#21. $S = \left\{2^n + \frac{1}{k} : n,k \in J\right\}$

Claim: 2^n, $n \in J$ is an accumulation point of S.

Justification: The numbers $2^n + \frac{1}{k}$ converges to 2^n as $k \to \infty$ because $\frac{1}{k} \to 0$ as $k \to \infty$.

Also, if the n is allowed to change, the 2^n will grow without bound.