HW2 - Solutions

§0.5 #41. Let $0<a<b$.

Show: $0<a^2<b^2$

Proof: First, from $0<a$ if we multiply by a (positive) we get $0<a^2$.

Next, if $a<b$, multiplying by a gives: $a^2<ab$.

and multiplying by b gives $ab<b^2$.

Then $a^2<ab<b^2$ so $a^2<b^2$.

Therefore $0<a^2<b^2$. $\boxed{\Sigma}$

Show: $0<\sqrt{a}<\sqrt{b}$.

Proof 1: By contradiction.

Suppose $\sqrt{a} \geq \sqrt{b}$. when $a<b$.

Applying the first part of this problem,

$\sqrt{b} \leq \sqrt{a} \implies (\sqrt{b})^2 \leq (\sqrt{a})^2 \iff b \leq a$.

But we assumed $a<b$, so we have a contradiction.

$\therefore \sqrt{a}<\sqrt{b}$.

(We also have $a>0 \implies \sqrt{a}>0$ (by def'n.).)

So $0<\sqrt{a}<\sqrt{b}$. $\boxed{\Sigma}$

Proof 2: Again, we focus on $a<b \implies \sqrt{a}<\sqrt{b}$.

$$\begin{aligned} a<b &\iff b-a>0 \\ &\iff (\sqrt{b})^2-(\sqrt{a})^2>0 \\ &\iff (\sqrt{b}-\sqrt{a})(\sqrt{b}+\sqrt{a})>0. \end{aligned}$$

Now the product of 2 real numbers is positive iff the two terms have the same sign.

We know $\sqrt{b}+\sqrt{a}>0$, so $\sqrt{b}-\sqrt{a}>0$ as well.

But, $\sqrt{b}-\sqrt{a}>0$ is the same as $\sqrt{b}>\sqrt{a}$ which is $\sqrt{a}<\sqrt{b}$. $\boxed{\Sigma}$

§1.1 #2. Let $x, y \in \mathbb{R}$.

Claim: $\exists$ nbhd P of x & nbhd Q of y s.t. $P \cap Q = \phi$.

Proof: Assume $y > x$, and choose $\varepsilon = \frac{1}{2}(y-x)$.

Define $P = (x-\varepsilon, x+\varepsilon)$ and $Q = (y-\varepsilon, y+\varepsilon)$.

P is a neighborhood of x & Q is a neighborhood of y.

To see that $P \cap Q = \phi$, we will show that

$$x+\varepsilon \leq y-\varepsilon.$$

Now: $x+\varepsilon = x+\frac{1}{2}(y-x) = \frac{1}{2}(y+x)$.

$y-\varepsilon = y-\frac{1}{2}(y-x) = \frac{1}{2}(y+x)$.

so $x+\varepsilon = y-\varepsilon$ and so $P \cap Q = \phi$. $\boxtimes$

#4. $a_n = \dfrac{3n+7}{n} = 3+\dfrac{7}{n}$.

Clearly, $a_n = 3+\frac{7}{n} > 3$ for all n (because $\frac{7}{n} > 0$).

Also, $a_n \leq 3+7 = 10$ for all n because $\frac{7}{n} \leq 7$ for all $n \in J$.

So, $\inf a_n = 3$ and $\sup a_n = 10$.

#5. $a_n = (-1)^n$ is bounded by $+1$ & -1 & does not converge.

Another example is $a_n = \sin n$, or $a_n = \sin\left(\frac{n\pi}{4}\right)$, or ....

#6.(d). The sequence $\left\{\dfrac{3n}{2n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{3}{2}$.

~~To prove this, let $\varepsilon > 0$ be given.~~

Before proving this claim let's do some scratch work to determine an appropriate N so that

$$\left|\frac{3n}{2n+1}-\frac{3}{2}\right| < \varepsilon \quad \text{for all } n \geq N.$$

Now, $\left|\dfrac{3n}{2n+1}-\dfrac{3}{2}\right| = \left|\dfrac{6n-3(2n+1)}{2(2n+1)}\right| = \left|\dfrac{-3}{4n+2}\right| = \dfrac{3}{4n+2}$

so $\left|\dfrac{3n}{2n+1}-\dfrac{3}{2}\right| < \varepsilon \iff \dfrac{3}{4n+2} < \varepsilon \iff \dfrac{3}{\varepsilon} < 4n+2$

$\iff \dfrac{3}{\varepsilon}-2 < 4n \iff \dfrac{3}{4\varepsilon}-\dfrac{1}{2} < n.$

Let's now give the proof.

Let $\varepsilon > 0$ be given.
Choose N to be the first positive integer strictly larger than $\frac{3}{4\varepsilon} - \frac{1}{2}$. (Note: if $\varepsilon > 0$ is large enough, $\frac{3}{4\varepsilon} - \frac{1}{2} < 0$. This is not a problem, but we have to make sure $N > 0$.)

For any $n \geq N$:

$$\left| \frac{3n}{2n+1} - \frac{3}{2} \right| = \frac{3}{4n+2} \overset{n \geq N}{\leq} \frac{3}{4N+2} \overset{N > \frac{3}{4\varepsilon} - \frac{1}{2}}{<} \frac{3}{4\left(\frac{3}{4\varepsilon} - \frac{1}{2}\right) + 2} = \frac{3}{\frac{3}{\varepsilon} - 2 + 2} = \frac{3}{3/\varepsilon} = \varepsilon.$$

so $\left| \frac{3n}{2n+1} - \frac{3}{2} \right| < \varepsilon$

Because this choice of N works for all choices of $\varepsilon > 0$, the convergence of $\left\{ \frac{3n}{2n+1} \right\}_{n=1}^{\infty}$ to $\frac{3}{2}$ has been proven. $\boxed{\Sigma}$

#10. Claim: if $\{a_n\}_{n=1}^{\infty}$ converges to A, then $\{|a_n|\}_{n=1}^{\infty}$ converges to $|A|$.

Proof: Let $\varepsilon > 0$ be given. Let $N \in \mathbb{J}$ be s.t. $|a_n - A| < \varepsilon \ \forall n \geq N$.

Case 1: $A = 0$
if $n \geq N$ then $\big| |a_n| - |A| \big| = \big| |a_n| - 0 \big| = \big| |a_n| \big|$
$= |a_n| = |a_n - 0| = |a_n - A| < \varepsilon$.

To see the converse is false, consider $a_n = (-1)^n$. The sequence $\{|a_n|\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$ converges to 1 but $\{a_n\}_{n=1}^{\infty}$ does not converge.

Case 2: $A > 0$
All but a finite number of the a_n must be positive.
(in fact, they have to be in $(0, 2A)$, for example)
Let M be the ~~first~~ first index with $a_n > 0 \ \forall n \geq M$
Define $N_0 = \max\{N, M\}$.
Then $\big| |a_n| - |A| \big| \underset{n \geq M}{=} |a_n - A| \underset{n \geq N}{<} \varepsilon \quad \forall n \geq N_0$.

Case 3: $A < 0$
Let M be the first index with $a_n < 0 \ \forall n \geq M$.
Define $N_0 = \max\{N, M\}$.
Then $\big| |a_n| - |A| \big| \underset{n \geq M}{=} |-a_n - (-A)| = |A - a_n| = |a_n - A| \overset{n \geq N}{<} \varepsilon \quad \forall n \geq N_0$

Since exactly one of these cases must be true, the argument holds for all $\varepsilon > 0$ and so $\{|a_n|\}_{n=1}^{\infty}$ conv. to $|A|$.