HW #11 - Solutions

#5. $h(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$

Note that $\lim\limits_{x\to 0} h(x) = 0 = h(0)$

so h is continuous at 0.

To find the derivative of h:

if $x \neq 0$: $h'(x) = x^3 \cos \frac{1}{x} \cdot \left(\frac{-1}{x^2}\right) + 3x^2 \sin \frac{1}{x} = -x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x}$.

if $x = 0$: $h'(0) = \lim\limits_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim\limits_{x\to 0} \frac{x^3 \sin \frac{1}{x} - 0}{x - 0} = \lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$

so $h'(x) = \begin{cases} -x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$, i.e. h is differentiable on $\mathbb{R}$.

To see if h' is continuous at 0: $\lim\limits_{x\to 0} h'(x) = \lim\limits_{x\to 0} -x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x} = 0 = h'(0)$.

To investigate the differentiability of h' at 0:

$$\lim_{x\to 0} \frac{h'(x) - h'(0)}{x - 0} = \lim_{x\to 0} \frac{-x \cos \frac{1}{x} + 3x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \Big(-\underbrace{\cos \tfrac{1}{x}}_{\substack{\text{bdd.}\\ \text{but does}\\ \text{not converge.}}} + \overbrace{3x}^{0} \underbrace{\sin \tfrac{1}{x}}_{\text{bdd.}}\Big)$$

so h' is not differentiable at $x = 0$.

#11. Write $f(x) = g(h(x))$ with $g(x) = \sqrt{x}$ ($D' = (0,\infty)$) and $h(x) = 2x^2 - 3x + 6$ (with $D = (0,1)$). Clearly, g is differentiable on $(0,\infty)$ and h is differentiable on all of $\mathbb{R}$. To show that $f = g \circ h$ is differentiable on $(0,1)$ all we have to verify is that $h(D) \subset D'$, that is $h((0,1)) \subset (0,\infty)$. The maximum value of $h(x)$ for $x \in [0,1]$ is $h(0) = 6$; the minimum value of $h(x)$ is $h\left(\frac{3}{4}\right) = \frac{39}{8}$. Thus $h((0,1)) = \left[\frac{39}{8}, 6\right) \subset (0,\infty)$, and f is differentiable on $(0,1)$. To find the formula for f':

$$f'(x) = \frac{1}{2}\left(2x^2 - 3x + 6\right)^{-1/2} \cdot (4x - 3) = \frac{4x-3}{2\sqrt{2x^2 - 3x + 6}}.$$

#20. Given $f: [0,2] \to \mathbb{R}$ that is differentiable with $f(0)=0$, $f(1)=2$, and $f(2)=2$.

i. Applying the MVT on $[1,2]$ we see $\exists c_1 \in (1,2)$ s.t. $f'(c_1) = \frac{f(2)-f(1)}{2-1} = \frac{2-2}{1} = 0$.

ii. Applying the MVT on $[0,1]$ we see $\exists c_2 \in (0,1)$ s.t. $f'(c_2) = \frac{f(1)-f(0)}{1-0} = \frac{2}{1} = 2$.

iii. ~~Applying the MVT on $[0,2]$ we see $\exists c_3 \in (0,2)$ s.t. $f'(c_3) = \frac{f(2)-f(0)}{2-0} = \frac{2}{2} = 1$.~~

By Theorem 4.11, since $f'(c_1)=0$ & $f'(c_2)=2$ & $0<\frac{3}{2}<2$, $\exists c_3$ between c_1 & c_2 with $f'(c_3)=\frac{3}{2}$.

#21. Let $f: [0,1] \to \mathbb{R}$ and $g: [0,1] \to \mathbb{R}$ be differentiable with $f(0)=g(0)$, and $f'(x) > g'(x)$ for all $x \in [0,1]$.

Claim: $f(x) > g(x)$ for all $x \in [0,1]$.

Proof: Define $h(x) = f(x) - g(x)$. for all $x \in [0,1]$.

Let $x \in [0,1]$ be given.

By the MVT $\exists c \in (0,x)$ such that $\frac{h(x)-h(0)}{x-0} = h'(c)$.

That is:
$$\frac{(f(x)-g(x)) - \underbrace{(f(0)-g(0))}_{=0}}{x-0} = (f'(c)-g'(c))$$

$$f(x)-g(x) = \underbrace{x}_{\in(0,1)} \underbrace{(f'(c)-g'(c))}_{>0} > 0$$

So $f(x) > g(x)$.

Since $x \in [0,1]$ is arbitrary, $f(x) > g(x)$ $\forall x \in [0,1]$. $\blacksquare$

Note: This says that if 2 cars are at the same place at some instant and that one car is always moving faster, then it always is ahead of the other car. Pretty obvious.

#33.
$$\lim_{x\to 0} \frac{x^2 \sin x}{\sin x - x\cos x} \overset{l'H}{=} \lim_{x\to 0} \frac{x^2\cos x + 2x\sin x}{\cos x - (\cos x - x\sin x)}$$
because $x^2 \sin x \to 0$ & $\sin x - x\cos x \to 0$

$$= \lim_{x\to 0} \frac{x^2\cos x + 2x\sin x}{x\sin x}$$

$$= \lim_{x\to 0} \frac{x\cos x + 2\sin x}{\sin x}$$
because $x \neq 0$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{\cos x - x\sin x + 2\cos x}{\cos x}$$
because $x\cos x + 2\sin x \to$ & $\sin x \to 0$.

$$= \lim_{x\to 0} \frac{3\cos x - x\sin x}{\cos x} = \frac{3}{1} = 3.$$