HW 10 - Solution

#41. To apply Bolzano's Theorem to this problem ($xe^x = 1$) define $f(x) = xe^x - 1$. Note that f is continuous on $\mathbb{R}$. From $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$, Bolzano's Theorem assures us that there is at least one number $c \in (0,1)$ with $f(c) = 0$, that is a solution to $xe^x = 1$.

#42. Let $f(x) = x^3 - 6x^2 + 2.826$.
This function is continuous on $\mathbb{R}$.
From $f(0) = 2.826 > 0$ and $f(1) = 1 - 6 + 2.826 = -2.174 < 0$ Bolzano's Theorem tells us that there must be at least one solution to $f(x) = 0$ in the interval $(0,1)$.
Moreover, $f(-1) = -1 - 6 + 2.826 = -4.174 < 0$ so Bolzano's Theorem also tells us that there must be at least one solution to $f(x) = 0$ in $(-1, 0)$.

#43. Let $f : [a,b] \to \mathbb{R}$ be continuous.
Suppose y is any number with $f(b) \le y \le f(a)$.
If $y = f(b)$ then choose $c = b$ (so $f(c) = f(b) = y$).
If $y = f(a)$ then choose $c = a$ (so $f(c) = f(a) = y$).
If $f(b) < y < f(a)$, ~~then~~ define $g(x) = f(x) - y$.
Then $g : [a,b] \to \mathbb{R}$ is continuous with
$g(a) = f(a) - y > 0$ and $g(b) = f(b) - y < 0$.
Bolzano's Theorem then tells us that there is at least one number c in (a,b) with $g(c) = 0$, that is $f(c) = y$.
Since the first two cases have $c = a$ and $c = b$ our final conclusion is that there is at least one number $c \in [a,b]$ with $f(c) = y$ (for any value of y with $f(b) \le y \le f(a)$). $\boxed{\Xi}$