HW #1 – Solutions

§0.1 #6. Prove: if $A \subset B$, then $(C \setminus B) \subset (C \setminus A)$.

Proof: Assume $A \subset B$.

Let $x \in C \setminus B$. Then $x \in C$ and $x \notin B$.

Because $A \subset B$, $x \notin B$ means $x \notin A$.

Thus $x \in C$ and $x \notin A$, which means $x \in C \setminus A$.

$\therefore (C \setminus B) \subset (C \setminus A)$. ∎

The converse is false, as can be illustrated by this counterexample.

$A = \{a\}$, $B = \{b\}$, $C = \{b\}$

$C \setminus B = \emptyset$ $\quad C \setminus A = \{b\}$ so $(C \setminus B) \subset (C \setminus A)$ but $A \not\subset B$.

#10. a) $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$.

b) $\bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R}$.

c) $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$.

d) $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 2+\frac{1}{n}\right) = (-1,3)$.

§0.2 #14. $f(x) = \frac{x}{x+2}$ $\qquad \operatorname{dom} f = \mathbb{R} \setminus \{-2\}$.

To find $\operatorname{im} f$ we ask for what values y is $f(x) = y$.

To answer this examine
$$\frac{x}{x+2} = y \iff x = y(x+2) = xy + 2y$$
$$\iff x - xy = 2y$$
$$\iff x(1-y) = 2y$$
$$\iff x = \frac{2y}{1-y}.$$

For any value of y, except $y = 1$, we see that there is a value of x that has $f(x) = \frac{x}{x+2} = y$. Thus, $\operatorname{im} f = \mathbb{R} \setminus \{1\}$.

To see that f is injective, suppose $f(u) = f(v)$. Then
$$\frac{u}{u+2} = \frac{v}{v+2} \iff u(v+2) = v(u+2) \iff uv + 2u = uv + 2v$$
$$\iff 2u = 2v \iff u = v.$$

Thus f is injective.

The inverse of f is $f^{-1} : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by $f^{-1}(x) = \frac{2x}{1-x}$.

§0.3 #20. Claim: For all $n \in J$, $1+3+5+\cdots+(2n-1)=n^2$.

Proof: by induction.

Let $P(n)$ be the statement that $1+3+\cdots+(2n-1)=n^2$.

Base: $n=1$: $1+3+5+\cdots+(2n-1)=1$ & $n^2=1$ so $P(1)$ is true.

Inductive step: Suppose $P(k)$ is true for some $k \in J$.

To show $P(k+1)$ is true we begin with $P(k)$:

$$1+3+5+\cdots+(2k-1)=k^2$$

Adding $2(k+1)-1=2k+1$ to both sides of this eqn:

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)=k^2+2k+1$$
$$=(k+1)^2.$$

so $P(k+1)$ is true.

By the Principle of Math. Induction, $P(n)$ is true for all $n \in J$. $\boxtimes$

#29. Claim: if $f(0)=7$, $f(1)=4$, $f(n)=6f(n-2)-f(n-1)$ for all $n \geq 2$. $(n \in \mathbb{Z})$

then $f(n)=5\cdot 2^n+2(-3)^n$ for all $n \in \mathbb{Z}$, $n \geq 0$.

Proof: Let $P(n)$ be the statement that $f(k)=5\cdot 2^k+2(-3)^k$

for $k=0,1,\ldots,n$.

Then $P(1)$ is true because $f(0)=5\cdot 2^0+2(-3)^0=5+2=7$

and $f(1)=5\cdot 2^1+2(-3)^1=10-6=4$.

Suppose $P(n)$ is true, so $f(k)=5\cdot 2^k+2\cdot(-3)^k$ for $k=0,1,\ldots,n$.

To show $P(n+1)$ is true it suffices to show that

$$f(n+1)=5\cdot 2^{n+1}+2(-3)^{n+1}.$$

Now, by definition of f,

$$f(n+1)=6f(n-1)-f(n)$$

The formula holds for $n-1$ & n so we have

$$f(n+1)=6\left(5\cdot 2^{n-1}+2(-3)^{n-1}\right)-\left(5\cdot 2^n+2(-3)^n\right)$$
$$=30\cdot 2^{n-1}-5\cdot 2^n+12(-3)^{n-1}-2(-3)^n$$
$$=15\cdot 2^n-5\cdot 2^n-4(-3)^n-2(-3)^n$$
$$=10\cdot 2^n-6(-3)^n$$
$$=5\cdot 2^{n+1}+2(-3)^{n+1}.$$

So $P(n+1)$ is true.

By the Principle of Math. Induction, $P(n)$ is true for all $n \in \mathbb{Z}, n \geq$

$\therefore f(n)=5\cdot 2^n+2(-3)^n$ for all $n \geq 0$, $n \in \mathbb{Z}$. $\boxtimes$

§0.4 #31. $f(n)=\begin{cases} n & n \text{ odd} \\ -(n-1) & n \text{ even.} \end{cases}$