Exam 2 Solution Key

1. a. Let $f: D \to \mathbb{R}$ with x_0 an accumulation point of D.
Then, ~~limit~~ f has a limit L at x_0 iff for every $\varepsilon > 0$
there exists a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ and $x \in D$
then $|f(x) - L| < \varepsilon$.

b. Suppose $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$, x_0 an accumulation point of D, and f and g have limits at x_0. If $f(x) \le g(x)$ for all $x \in D$, then

$$\lim_{x \to x_0} f(x) \le \lim_{x \to x_0} g(x).$$

c. Let $f: E \to \mathbb{R}$ with $x_0 \in E$ and x_0 an accumulation point of E.
Then (i) through (iii) are equivalent:
i. f is continuous at x_0
ii. $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$.
iii. every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E$ $\forall n$ and $\lim_{n \to \infty} x_n = x_0$ has $\lim_{n \to \infty} f(x_n) = f(x_0)$.

d. A set $E \subset \mathbb{R}$ is closed iff
every accumulation point of E is in E.

e. A set $E \subset \mathbb{R}$ is open iff
for every $x \in E$ there is a neighborhood Q of x with $Q \subset E$.

f. Heine-Borel Theorem:
A set $E \subset \mathbb{R}$ is compact iff E is closed and bounded.

2. Let $\varepsilon = \frac{1}{2}$. Since f is continuous at $x_0 = 0$, $\exists\, \delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon = \frac{1}{2}$. Notice that $|f(x) - f(0)| < \frac{1}{2}$ is equivalent to $|f(x) + 1| < \frac{1}{2}$ so $-\frac{1}{2} < f(x) + 1 < \frac{1}{2}$ and then $-\frac{3}{2} < f(x) < -\frac{1}{2}$. That is if $x \in Q = (-\delta, \delta)$ then $f(x) < -\frac{1}{2}$. (The set Q is the neighborhood of 0.)

3. $f(x) = \dfrac{1-\sqrt{1+x}}{x}$ for $x \in (0,1)$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\sqrt{1+x}}{x} \cdot \frac{1+\sqrt{1+x}}{1+\sqrt{1+x}} = \lim_{x\to 0} \frac{1-(1+x)}{x(1+\sqrt{1+x})} = \lim_{x\to 0} \frac{-x}{x(1+\sqrt{1+x})}$$

$$= \lim_{x\to 0} \frac{-1}{1+\sqrt{1+x}} = -\frac{1}{2}.$$

Thus, if we define $f(0) = -\frac{1}{2}$ then $\lim_{x\to 0} f(x) = -\frac{1}{2} = f(0)$

and so f is continuous at $x=0$.

4. (a) To show f is not continuous at $x=2$,

let $\{x_n\}_{n=1}^{\infty}$ be a sequence of irrational numbers with $\lim_{n\to\infty} x_n = 2$.

Then $f(x_n) = \frac{1}{2}(x_n^2+1)$ for each $n = 1, 2, \ldots$

and so $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{2}(x_n^2+1) = \frac{1}{2}\left((\lim_{n\to\infty} x_n)^2+1\right) = \frac{1}{2}(2^2+1) = \frac{5}{2}$

which is not equal to $f(2) = 2^2 = 4$.

$\therefore$ f is not continuous at $x=2$.

(b). To have f continuous at x_0 all ~~must~~ rational sequences $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = x_0$ ~~must~~ have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n)^2 = x_0^2$.

and all irrational sequences $\{y_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} y_n = x_0$ have

$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \frac{1}{2}(y_n^2+1) = \frac{1}{2}(x_0^2+1)$ with

$$x_0^2 = \frac{1}{2}(x_0^2+1) \iff 2x_0^2 = x_0^2+1$$
$$\iff x_0^2 = 1$$
$$\iff x_0 = \pm 1.$$

5. (a) i. any set $[a,b)$ or $(a,b]$ with $a<b$. is neither open nor closed.

ii. the sets ϕ and $\mathbb{R}$ are both open & closed.

iii. any set $[a,\infty)$ or $(-\infty, a]$ is closed and unbounded (& so is not compact).

(b). Typical examples are $\{G_n\} = \{(0,n)\}_{n=1}^{\infty}$ or $\{G_n\} = \{(\frac{1}{n}, n)\}_{n=1}^{\infty}$

or $\{G_n\} = \{(\frac{1}{n}, \infty)\}$ but there are many other possible answers.