# Exam 1 – Solutions

#1. a) $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, n\right) = (-1,1) \cup \left(-\frac{1}{2},2\right) \cup \left(-\frac{1}{3},3\right) \cup \cdots$
$= (-1, \infty)$.

b) $\bigcap_{n=1}^{\infty} \left(-1+\frac{1}{n}, 1+\frac{1}{n}\right) = (0,2) \cap \left(-\frac{1}{2}, \frac{3}{2}\right) \cap \left(-\frac{2}{3}, \frac{4}{3}\right) \cap \cdots$
$= (0, 1]$.

#2. Show: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof: First, let $x \in A \setminus (B \cap C)$.

Then $x \in A$ and $x \notin B \cap C$.

Now $x \notin B \cap C$ means $x \notin B$ or $x \notin C$.

Thus, $x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$.

That is, $x \in (A \setminus B) \cup (A \setminus C)$.

$\therefore \; A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$.

Next, let $x \in (A \setminus B) \cup (A \setminus C)$.

Then $x \in A \setminus B$ or $x \in A \setminus C$.

Now $x \in A \setminus B$ means $x \in A$ and $x \notin B$.

and $x \in A \setminus C$ means $x \in A$ and $x \notin C$.

Thus we see that $x \in A$ and either $x \notin B$ or $x \notin C$.

but this means $x \in A$ and $x \notin B \cap C$, and so $x \in A \setminus (B \cap C)$.

$\therefore \; (A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C)$.

Therefore these two sets are equal:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C). \quad \blacksquare$$

1. (10 points) Describe each of the following sets as the empty set, as $R$, or in interval notation, as appropriate:

   (a) $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, n\right)$

   (b) $\bigcap_{n=1}^{\infty} \left(-1+\frac{1}{n}, 1+\frac{1}{n}\right)$

2. (10 points) Let $A$, $B$, and $C$ be sets. Show that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

3. (32 points) Fill in each blank with the word or phrase or equation that completes the thought.

   (a) A *sequence* is a function whose domain is the set of $J$, natural numbers, positive integers.

   (b) A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number $A$ iff for each $\epsilon > 0$ $\exists N \in J$ s.t. $|a_n - A| < \varepsilon \ \forall n \geq N$.

   (c) A sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy iff for each $\epsilon > 0$ $\exists N \in J$ s.t. $|a_n - a_m| < \varepsilon \ \forall m, n \geq N$

   (d) Let $S$ be the set of real numbers. A real number $A$ is an accumulation point of $S$ iff every neighborhood of $A$ contains an infinite number of elements of $S$.

   (e) Bolzano-Weierstrass Theorem: Every bounded infinite set of real numbers has at least one accumulation point.

   (f) If $\{a_n\}_{n=1}^{\infty}$ converges to $A$ and $\{b_n\}_{n=1}^{\infty}$ converges to $B$, with $b_n \neq 0 \ \forall n \in J$ & $B \neq 0$, then $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ converges to $A/B$.

   (g) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $n_1 < n_2 < n_3 < \cdots$. The sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$.

   (h) Theorem: A monotone sequence is convergent iff it is bounded.

4. (16 points) Prove: Every Cauchy sequence is bounded.

5. (16 points) Use the definition of convergence to prove that $\left\{\frac{3n}{2n-1}\right\}_{n=1}^{\infty}$ converges.

6. (16 points) Prove: If $\{a_n\}_{n=1}^{\infty}$ converges to $A$ and $\{b_n\}_{n=1}^{\infty}$ converges to $B$, then $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A + B$

4. Claim: Every Cauchy sequence is bounded.

Proof: Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence.

Let $\varepsilon = 1$.

Because $\{a_n\}_{n=1}^{\infty}$ is Cauchy, $\exists N \in \mathbb{J}$ s.t. $|a_n - a_m| < 1 \;\forall m,n \geq N$.

In particular, with $m = N$: $|a_n - a_N| < 1 \;\forall n \geq N$.

Observe that $|a_n - a_N| < 1$ means $-1 + a_N < a_n < 1 + a_N \;\forall n \geq N$.

Let $M = \max\{a_1, a_2, \ldots, a_{N-1}, 1 + a_N\}$

and $L = \min\{a_1, a_2, \ldots, a_{N-1}, -1 + a_N\}$

Then $L \leq a_n \leq M$ for all $n \in \mathbb{J}$, i.e. $\{a_n\}_{n=1}^{\infty}$ is bounded. $\boxed{}$

5. First, this sequence converges to $3/2$.

To prepare for the proof:

$$|a_n - L| = \left|\frac{3n}{2n-1} - \frac{3}{2}\right| = \left|\frac{2\cdot 3n - 3(2n-1)}{2(2n-1)}\right| = \left|\frac{3}{2(2n-1)}\right| = \frac{3}{4n-2} < \varepsilon$$

provided $\frac{4n-2}{3} > \frac{1}{\varepsilon} \iff 4n-2 > \frac{3}{\varepsilon} \iff 4n > \frac{3}{\varepsilon} + 2 \iff n > \frac{3}{4\varepsilon} + \frac{1}{2}$.

Proof: Let $\varepsilon > 0$ be given.

Choose $N$ to be the first pos. integer larger than $\frac{3}{4\varepsilon} + \frac{1}{2}$.

When $n \geq N$:

$$|a_n - L| = \ldots = \frac{3}{4n-2} \leq \frac{3}{4N-2} < \frac{3}{4\left(\frac{3}{4\varepsilon} + \frac{1}{2}\right) - 2} = \frac{3}{\frac{3}{\varepsilon} + 2 - 2}$$

$$= \frac{3}{3/\varepsilon} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have shown that

$\forall \varepsilon > 0 \;\exists N \in \mathbb{J}$ s.t. $|a_n - L| < \varepsilon \;\forall n \geq N$.

Hence, $\{a_n\}_{n=1}^{\infty}$ converges to $\frac{3}{2}$. $\boxed{}$

6. Claim: if $\{a_n\}_{n=1}^{\infty}$ converges to $A$ and $\{b_n\}_{n=1}^{\infty}$ converges to $B$ then $\{a_n+b_n\}_{n=1}^{\infty}$ converges to $A+B$.

Proof: Let $\varepsilon > 0$ be given.

Because $\{a_n\}_{n=1}^{\infty}$ converges to $A$,

$\exists N_1 \in J$ s.t. $|a_n - A| < \frac{\varepsilon}{2}$ $\forall n \geq N_1$.

Because $\{b_n\}_{n=1}^{\infty}$ converges to $B$,

$\exists N_2 \in J$ s.t. $|b_n - B| < \frac{\varepsilon}{2}$ $\forall n \geq N_2$

Choose $N = \max\{N_1, N_2\}$.

Then, for all $n \geq N$:

$$\begin{aligned}
|(a_n+b_n) - (A+B)| &= |a_n - A + b_n - B| \\
&\leq |a_n - A| + |b_n - B| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{(because } n \geq N_1 \text{ \& } n \geq N_2\text{)}. \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$\forall \varepsilon > 0$ $\exists N \in J$ s.t. $|(a_n - b_n) - (A+B)| < \varepsilon$ $\forall n \geq N$.

That is, $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $A+B$. $\boxtimes$