Ch 8: The Integral Theorems of Vector Analysis

8.a. Regions

Example: $D \stackrel{\text{def}}{=} \{(x,y)\colon x^2+y^2 \leqslant 1\} \subset \mathbb{R}^2$ is a GREEN'S REGION

$\delta D = \{(x,y)\colon x^2+y^2 = 1\} \subset D$ is the boundary of D

$\delta D^+ = \langle \cos t, \sin t\rangle \quad 0 \leqslant t \leqslant 2\pi$ is a (counterclockwise) parameterization of δD

$\vec{n} = \langle x, y\rangle$ is the outward unit normal to δD .

▶ Let D be a Green's region

$\vec{F}(x,y) = \langle P(x,y)\ ,\ Q(x,y)\rangle$ be C^1 on D .

8.a.1. Green's Theorem for $D \subset \mathbb{R}^2$

Then
$$\int_{\delta D^+} \vec{F} \bullet d\vec{s} = \iint_D \left(\text{curl } \vec{F} \bullet \vec{k}\right) dA$$
$$\qquad\| \qquad\qquad\qquad\qquad \|$$
i.e.
$$\int_{\delta D^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$\rightsquigarrow$ so
$$\int_{\delta D^+} \langle -y\ ,\ x\rangle \bullet d\vec{s} = 2\ \text{(area of } D) .$$

8.a.2. Divergence Theorem for $D \subset \mathbb{R}^2$

Then
$$\int_{\delta D^+} \left(\vec{F} \bullet \vec{n}\right) ds = \iint_D \left(\text{ div } \vec{F}\ \right) dA .$$

Example: $\mathcal{S} \stackrel{\text{def}}{=} \{(x,y,z)\colon x^2+y^2+z^2 = 1 \text{ and } z \geqslant 0\} \subset \mathbb{R}^3$ is a STOKES' SURFACE

$\delta\mathcal{S} = \{(x,y,0)\colon x^2+y^2 = 1\} \subset \mathcal{S}$ is the boundary of $\mathcal{S}$

$\delta\mathcal{S}^+ = \langle \cos t, \sin t, 0\rangle \quad 0 \leqslant t \leqslant 2\pi$ is a (cc-wise) parameterization of $\delta\mathcal{S}$.

8.a.3. Stokes' Theorem for $\mathcal{S} \subset \mathbb{R}^3$

If $\mathcal{S}$ is a Stokes' Surface

$\vec{F}\colon \mathcal{S} \to \mathbb{R}^3$ is C^1 on $\mathcal{S}$

then
$$\int_{\delta\mathcal{S}^+} \vec{F} \bullet d\vec{s} = \iint_{\mathcal{S}} \left(\text{ curl } \vec{F}\ \right) \bullet d\vec{S} .$$

Example: $\Omega \stackrel{\text{def}}{=}$ soup can (top & bottom included) + the soup is a GAUSS' REGION

$\delta\Omega =$ the soup can (top & bottom included) is the boundary of $\mathcal{S}$.

8.a.4. Gauss' Divergence Theorem for $\Omega \subset \mathbb{R}^3$

If Ω is a Gauss' Surface

$\vec{F}\colon \Omega \to \mathbb{R}^3$ is C^1 on Ω

then
$$\iint_{\delta\Omega} \vec{F} \bullet d\vec{S} = \iiint_{\Omega} \left(\text{ div } \vec{F}\ \right) dV .$$

8.b. $\boxed{\text{Definitions of Green's Region, Stokes' Surface, Gauss' Region}}$ see **7.a** for path defs

8.b.1. <u>Elementary Green's Region D</u> and corresponding items:

$D \subset \mathbb{R}^2$ is an elementary region (**5.2**)

δD is the boundary of D (**6.2**)

$\delta D^+ \quad = \quad \langle x(t)\,,\, y(t)\rangle$ is a piecewise-smooth (simple closed) path that parameterizes δD in the positive (ie. counterclockwise) direction

$\vec{n} \quad = \quad \dfrac{\langle\, y'(t)\,,\, -x'(t)\,\rangle}{\|\,\langle\, y'(t)\,,\, -x'(t)\,\rangle\,\|}$ is the outward unit normal vector to δD .

$\rightsquigarrow$ note that if you walk along δD^+, then D is to your left

$\rightsquigarrow$ a region that can be divided into a finite number of Elementary Green's Regions, with their shared δD's "oppositely oriented," is a Green's Region (see p. 471)

8.b.2. <u>Elementary Stokes' Surface $\mathcal{S}$</u> and corresponding items:

Given $\underset{\mathbb{R}^2}{\underset{\cap}{D}}{}^2 \xrightarrow[\text{onto}]{\vec{\Phi}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$

where $\vec{\Phi}$ is a parameterization of the surface $\mathcal{S}$ (that is more-than-NICE (**7.d.3**), namely:)

D is an Elementary Green's region

Φ is C^2 on D and 1-to-1 on (all of) D

$\mathcal{S}$ is smooth and oriented.

So $\delta D^+ \overset{\textbf{8.b.1}}{=\!=} \langle u(t)\,,\, v(t)\rangle$ is a piecewise-smooth (oriented simple closed) path that parameterizes δD in the positive (ie. counterclockwise) direction .

Let: $\delta S \overset{\text{def}}{=} \vec{\Phi}(\delta D)$

$\delta S^+ \overset{\text{def}}{=} \vec{\Phi}(\delta D^+)$.

Then $\delta\mathcal{S}$ is the BOUNDARY of $\mathcal{S}$

δS^+ is a (oriented simple closed) path that parameterizes δS .

also need if you walk along δS^+, standing on the positive side of $\mathcal{S}$, then $\mathcal{S}$ is to your left .

$\rightsquigarrow$ a surface that can be divided into a finite number of Elementary Stokes' Surfaces, with their shared δS's "oppositely oriented," is a Stokes' Surface

8.b.3. <u>Gauss' Region Ω</u> and corresponding items:

$\Omega \subset \mathbb{R}^3$ is a simple region (**5.3**)

$\delta\Omega \subset \mathbb{R}^3$ is the boundary of Ω (**6.2**)

$\delta\Omega$ is piecewise NICE (**7.d.3**), boundaries "oppositely oriented"

$\delta\Omega$ is oriented by the outward normal vectors .

8.c. $\boxed{\text{Conservative Fields}}$

8.c.1. Theorem (compare with **4.4**)

If $\vec{F} : \mathbb{D}^3 \to \mathbb{R}^3$ is a C^1 vector field

where $\mathbb{D}^3 =$ all of $\mathbb{R}^3$ expect for possibly a finite number of points,

then the following are equivalent:

(1) $\vec{F} = \nabla f$ for some scalar function $f : \mathbb{D}^3 \to \mathbb{R}$

(2) curl $F = \vec{0}$

(3) $\int_C \vec{F} \bullet d\vec{s} = 0$ for *each* oriented simple closed curve C

(4) $\int_{C_1} \vec{F} \bullet d\vec{s} = \int_{C_2} \vec{F} \bullet d\vec{s}$ for *each* pair of oriented simple curves C_1 & C_2 that both start at the same point and both end at the same point.

If $\vec{F}$ satisfies one (and hence all) of the above conditions,

then $\vec{F}$ is a GRADIENT VECTOR FIELD

$\vec{F}$ is a CONSERVATIVE VECTOR FIELD

f is the POTENTIAL of $\vec{F}$

$$f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

where $\vec{F} = \langle F_1, F_2, F_3 \rangle$.

7.c.6. Given $\vec{c}: [a,b] \to \mathbb{R}^3$ with $\vec{c}$ piecewise C^1

$f : \mathbb{R}^3 \to \mathbb{R}$ with $f \in C^2$

so $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle : \mathbb{R}^3 \to \mathbb{R}^3$

then $\int_{\vec{c}} \nabla f \bullet d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$.