Ch 7: Integrals over Paths and Surfaces

7.a. Paths Defs:

- ▷ A PATH is a (continuous) vector-valued function $\vec{c}\colon [a,b] \to \mathbb{R}^n$. Usually, $n = 2$ or 3.

- ▷ The CURVE $\mathcal{C}$ that is PARAMETERIZED by $\vec{c}$ is $\{\vec{c}(t)\colon a \leqslant t \leqslant b\}$.

- ▷ If $I \subset [a,b]$

  then $\vec{c}$ is 1-to-1 on I $\Leftrightarrow$ the curve $\{\vec{c}(t)\colon t \in I\}$ does not intersect itself.

- ▷ $\mathcal{C}$ is a CLOSED CURVE if $\vec{c}(a) = \vec{c}(b)$.

- ▷ $\mathcal{C}$ is a SMOOTH if it has a parameterization $\vec{c}$ where:

  (1) $\vec{c}$ is C^1 on $[a,b]$

  (2) $\vec{c}$ is 1-to-1 on $[a,b)$

  (3) $\vec{c}\,' \neq \vec{0}$ on $[a,b]$, *i.e.*, an object moving along $\mathcal{C}$ never comes to rest.

- ▷ $\mathcal{C}$ is a SIMPLE CURVE if it has a parameterization $\vec{c}$ where:

  (1) $\vec{c}$ is piecewise C^1 on $[a,b]$

  (2) $\vec{c}$ is 1-to-1 on $[a,b]$.

- ▷ $\mathcal{C}$ is a SIMPLE CLOSED CURVE if $\mathcal{C}$ is simple and closed.

- ▷ Given $[a,b] \xrightarrow[\text{onto}]{h} [a_1,b_1] \xrightarrow{\vec{c}} \mathbb{R}^n$

  with $h \in C^1$ and h is 1-to-1 on $[a,b]$

  $\vec{c}$ is piecewise C^1 on $[a_1,b_1]$

  then $\vec{p} \overset{\text{def}}{=} \vec{c} \circ h\colon [a,b] \to \mathbb{R}^n$ is a REPARAMETERIZATION of $\vec{c}$.

  Note: $\{\vec{c}(t)\colon a_1 \leqslant t \leqslant b_1\} = \{\vec{p}(t)\colon a \leqslant t \leqslant b\} = \mathcal{C}$.

  Since h is 1-to-1, there are 2 possibilities:

  either $h' > 0$ on $[a,b]$ $\Rightarrow$ $[a_1,b_1] = [h(a),h(b)]$ $\Rightarrow$ h PRESERVES ORIENTATION

  or $h' < 0$ on $[a,b]$ $\Rightarrow$ $[a_1,b_1] = [h(b),h(a)]$ $\Rightarrow$ h REVERSES ORIENTATION .

  Since $\vec{p}\,'(t) = \vec{c}\,'(h(t))\,h'(t)$

  one thinks of h as changing the speed at which an object moves along the curve $\mathcal{C}$.

  If $h(t) = a + b - t$

  then $\vec{p}(t) = \vec{c}(a+b-t) = \vec{c}_{\mathrm{op}}(t) =$ the opposite path to $\vec{c}$.

7.b. Path Integrals

7.b.1. Given $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$

with $\vec{c} \in C^1[a,b]$

f is a scalar field

$f \circ \vec{c}$ is continuous on $[a,b]$

then the PATH INTEGRAL of f over $\vec{c}$ is $\boxed{\int_{\vec{c}} f\,ds \stackrel{\text{def}}{=} \int_a^b f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,dt}$.

Note: If $\vec{c}$ is piecewise C^1 on $[a,b]$, we work piece-by-piece.

7.b.2. Key Idea:

Form a partition of $[a,b]$: $a = t_0 < t_1 < \ldots < t_n = b$,

with $\Delta t_i \equiv t_i - t_{i-1}$,

pick a selection $t_i^* \in [t_{i-1}, t_i]$.

Then
$$\begin{aligned}
\int_{\vec{c}} f\,ds &= \int_a^b [\,f(\vec{c}(t))\,\|\vec{c}\,'(t)\|\,]\,dt \\
&\approx \sum_{i=1}^n [\,f(\vec{c}(t_i^*))\,\|\vec{c}\,'(t_i^*)\|\,]\,\Delta t_i \\
&= \sum_{i=1}^n f(\vec{c}(t_i^*))\,[\,\|\vec{c}\,'(t_i^*)\|\,\Delta t_i\,] \\
&\approx \sum_{i=1}^n f(\vec{c}(t_i^*))\,\|\vec{c}(t_i) - \vec{c}(t_{i-1})\| \\
&\approx \sum_{i=1}^n f(\vec{c}(t_i^*))\,[\text{length of the curve } \vec{c}\,([t_{i-1}, t_i])\,] \,.
\end{aligned}$$

7.b.3. Examples:

▷ $\int_{\vec{c}} 1\,ds$ = length of the path $\vec{c}$.

▷ the AVERAGE VALUE of f along $\vec{c}$ is $\frac{\int_{\vec{c}} f\,ds}{\int_{\vec{c}} 1\,ds}$

e.g. $f(x,y,z)$ is the temperature at the point (x,y,z)

so average value of f along $\vec{c}$ = average temperature of f along $\vec{c}$.

▷ If $\vec{c}$ represents a wire in $\mathbb{R}^3$

with $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ the mass density (*e.g.*, grams/cm) of $\vec{c}$

then the total mass of the wire $= \int_{\vec{c}} \rho\,ds$

and the center of mass of the wire $= \left(\frac{\int_{\vec{c}} x\rho\,ds}{\int_{\vec{c}} \rho\,ds}\,,\ \frac{\int_{\vec{c}} y\rho\,ds}{\int_{\vec{c}} \rho\,ds}\,,\ \frac{\int_{\vec{c}} z\rho\,ds}{\int_{\vec{c}} \rho\,ds} \right)$.

▷ If $n = 2$

then $\int_{\vec{c}} f\,ds$ is the surface area of one side of the fence over $\vec{c}$ with height given by f .

7.b.4. Given **7.b.1** setting

$\vec{p}$, a reparameterization of $\vec{c}$

then $\int_{\vec{p}} f\,ds = \int_{\vec{c}} f\,ds$,

i.e. the value of a path integral does not depend on the parameterization of the path.

Note: This is true regardless of whether $\vec{p}$ preserves or reverses the orientation of $\vec{c}$.

7.c. Line Integrals

7.c.1. Given $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$

with $\vec{c} \in C^1[a,b]$

$\vec{F}$ is a vector field

$\vec{F} \circ \vec{c}$ is continuous on $[a,b]$

then the LINE INTEGRAL of $\vec{F}$ over $\vec{c}$ is $\boxed{\int_{\vec{c}} \vec{F} \bullet d\vec{s} \overset{\text{def}}{=} \int_a^b \vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\ dt}$.

Rmk: If $\vec{c}\,'(t) \neq \vec{0}$ on $[a,b]$

so $\vec{T}(t) \overset{\text{def}}{=} \vec{c}\,'(t) \,/\, \|\vec{c}\,'(t)\|$ is the unit tangent vector to $\vec{c}(t)$

then $\int_{\vec{c}} \vec{F} \bullet d\vec{s} \ = \ \int_a^b \Big(\vec{F}(\vec{c}(t)) \bullet \vec{T}(t) \Big) \ \|\vec{c}\,'(t)\| \ dt$.

7.c.2. Differential Form (a.k.a. tensor) Notation:

If $\vec{c}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$

then
$$\begin{aligned}
\int_{\vec{c}} \vec{F} \bullet d\vec{s} \ &= \ \int_a^b \langle F_1, F_2, F_3 \rangle \bullet \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} \rangle \ dt \\
&= \ \int_a^b \Big[F_1 \tfrac{dx}{dt} \ + \ F_2 \tfrac{dy}{dt} \ + \ F_3 \tfrac{dz}{dt} \Big] \ dt \\
&\overset{\text{notation}}{=} \int_{\vec{c}} \ F_1\, dx \ + \ F_2\, dy \ + \ F_3\, dz \ .
\end{aligned}$$

7.c.3. Key Idea: notation as in **7.b.2** and setting as in **7.c.1** :

Then
$$\begin{aligned}
\int_{\vec{c}} \vec{F} \bullet d\vec{s} \ &= \ \int_a^b \Big[\vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t) \Big] \ dt \\
&\approx \ \textstyle\sum_{i=1}^n \Big[\vec{F}(\vec{c}(t_i^*)) \bullet \vec{c}\,'(t_i^*) \Big] \ \Delta t_i \\
&\approx \ \textstyle\sum_{i=1}^n \ \vec{F}(\vec{c}(t_i^*)) \bullet \ [\vec{c}\,'(t_i^*)\, \Delta t_i\,] \\
&\approx \ \textstyle\sum_{i=1}^n \ \vec{F}(\vec{c}(t_i^*)) \bullet \ [\, \vec{c}\,(t_i) - \vec{c}\,(t_{i-1})] \ .
\end{aligned}$$

7.c.4. Example:

If $\vec{F}$ is a force field acting on a puffo

then $\int_{\vec{c}} \vec{F} \bullet d\vec{s}$ is the work done by $\vec{F}$ on the puffo as it moves along the path $\vec{c}$.

7.c.5. Given the **7.c.1** setting

$\vec{p}$, a reparameterization of $\vec{c}$.

If $\vec{p}$ preserves orientation then $\int_{\vec{p}} \vec{F} \bullet d\vec{s} \ = \ {}^{+} \int_{\vec{c}} \vec{F} \bullet d\vec{s}$.

If $\vec{p}$ reverses orientation then $\int_{\vec{p}} \vec{F} \bullet d\vec{s} \ = \ {}^{-} \int_{\vec{c}} \vec{F} \bullet d\vec{s}$.

7.c.6. Given $\vec{c} \colon [a,b] \to \mathbb{R}^3$ with $\vec{c}$ piecewise C^1

$f \colon \mathbb{R}^3 \to \mathbb{R}$ with $f \in C^1$

so $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \colon \mathbb{R}^3 \to \mathbb{R}^3$

then $\int_{\vec{c}} \nabla f \bullet d\vec{s} \ = \ f(\vec{c}(b)) - f(\vec{c}(a))$.

7.d. Surfaces

7.d.1. Given $\vec{\Phi}\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$

so $\vec{\Phi}(u,v) = (\, x\,(u,v)\,, y\,(u,v)\,, z\,(u,v)\,)$.

Def: $\vec{\Phi}$ is a PARAMETRIZED SURFACE

$\mathcal{S}$ is the SURFACE corresponding to $\vec{\Phi}$.

If $\vec{\Phi} \in C^1$

i.e. x and y and z are all in C^1

then $\mathcal{S}$ is a DIFFERENTIABLE (OR C^1) surface .

Rmk: Think of $\vec{\Phi}$ as twisting and bending the 2D region D^2 into a surface $\mathcal{S}$ sitting in 3D.

7.d.2. Key Idea:

Given $\vec{\Phi}\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a C^1 parametrized surface .

Fix $(u_0, v_0) \in D^2$

so $\vec{\Phi}(u_0, v_0) \equiv (x_0, y_0, z_0) \in \mathcal{S}$.

Then $\vec{c}_{u_0}(t) = \vec{\Phi}(u_0, t)$ is a path whose image lies on $\mathcal{S}$ and goes through (x_0, y_0, z_0) at $t = v_0$

and a tangent vector to $\vec{c}_{u_0}$ at (x_0, y_0, z_0) is $\overrightarrow{T_v}(u_0, v_0) \;\equiv\; \vec{c}\,'_{u_0}(v_0) \;=\; \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle\big|_{(u_0,v_0)}$

so $\overrightarrow{T_v}(u_0, v_0)$ is the instantaneous rate of change in $\vec{\Phi}$, in the direction of $u = u_0$, at (u_0, v_0) .

Lkws $\vec{c}_{v_0}(t) = \vec{\Phi}(t, v_0)$ is a path whose image lies on $\mathcal{S}$ and goes through (x_0, y_0, z_0) at $t = u_0$

and a tangent vector to $\vec{c}_{v_0}$ at (x_0, y_0, z_0) is: $\overrightarrow{T_u}(u_0, v_0) \;=\; \vec{c}\,'_{v_0}(u_0) \;=\; \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle\big|_{(u_0,v_0)}$

so $\overrightarrow{T_u}(u_0, v_0)$ is the instantaneous rate of change in $\vec{\Phi}$, in the direction of $v = v_0$, at (u_0, v_0) .

Let

$$\vec{n}_{(u_0,v_0)} \;=\; \overrightarrow{T_u}(u_0, v_0) \times \overrightarrow{T_v}(u_0, v_0) \;=\; \left| \begin{matrix} \vec{i} & \vec{i} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{matrix} \right|_{(u_0,v_0)} .$$

If $\vec{n}_{(u_0,v_0)} \neq \vec{0}$

then $\vec{n}_{(u_0,v_0)}$ is NORMAL to the surface $\mathcal{S}$ at (u_0, v_0)

and the equation of the TANGENT PLANE to the suface $\mathcal{S}$ at the point (x_0, y_0, z_0) is:

$$< x - x_0, y - y_0, z - z_0 > \;\bullet\; \vec{n}_{(u_0,v_0)} \;=\; \vec{0}\ .$$

Def: If $\vec{n}_{(u_0,v_0)} \neq \vec{0}$

then $\mathcal{S}$ is SMOOTH at (u_0, v_0) .

If $\vec{n}_{(u,v)} \neq \vec{0}$ for each $(u,v) \in D^2$

then $\mathcal{S}$ is SMOOTH .

Rmk: loosely speaking, a smooth surface has no corners or breaks.

7.d.3. Given $\vec{\Phi}\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a parameterized surface

with D^2 is an elementary region in uv-plane

$\vec{\Phi}$ is C^1 on D^2 and 1-to-1 on the interior of D^2

$\mathcal{S}$ is smooth, except at a finite number of points

then $\vec{\Phi}$ & $\mathcal{S}$ are <u>NICE</u>

Note: This is not standard terminology, but it is descriptive.

If $\vec{\Phi}$ & $\mathcal{S}$ are a (finite, disjoint) union of NICE $\vec{\Phi}_i$'s & $\mathcal{S}_i$'s

then $\vec{\Phi}$ & $\mathcal{S}$ are <u>PIECEWISE NICE</u> (pw-nice).

7.d.4. Have $\vec{\Phi}\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ a NICE parameterized surface

then

$$\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\| = \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2} \quad \text{where} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} .$$

7.d.5. Given $g\colon \underset{\mathbb{R}^2}{\underset{\cap}{D^2}} \longrightarrow \mathbb{R}$.

Form $\vec{\Phi}\colon D^2 \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\underset{\cap}{\mathcal{S}}}$ via $\vec{\Phi}(u,v) = (u,v,g(u,v))$.

If D^2 is an elementary region

$g \in C^1$

$(u_0, v_0) \in D^2$

then $\vec{\Phi}$ is nice

$P_0 \equiv (u_0, v_0, g(u_0, v_0)) \in \mathcal{S}$

$\vec{n} = \vec{n}\big|_{P_0} = \pm\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, {}^{-}1\rangle\big|_{P_0}$ is a downward(upward) normal vector to $\mathcal{S}$ at P_0

and

$$\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} = \frac{1}{\cos\measuredangle\vec{k}\vec{n}} = \frac{\|\vec{n}\|}{\vec{n} \bullet \vec{k}} .$$

7.e. Surface Integrals of Scalar–Valued Functions

7.e.1. Given $\underset{\mathbb{R}^2}{\overset{\cap}{D^2}} \xrightarrow[\text{onto}]{\vec{\Phi}} \underset{\mathbb{R}^3}{\overset{\cap}{\mathcal{S}}} \xrightarrow{f} \mathbb{R}$

with $\vec{\Phi}$ is NICE

f is continuous

then the INTEGRAL OF f OVER $\mathcal{S}$ is :

$$\iint_{\mathcal{S}} f\,dS \;=\; \iint_{D^2} f\left(\vec{\Phi}\,(u,v)\right)\;\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\;du\,dv\ .$$

7.e.2. Key Idea: using notation from **5.2** (see text pg 431&442)

Form a partition $\mathcal{R} = \{R_{ij}\ :\ R_{ij} \subset D^2\}$ of D^2 into rectangles

with mesh $\mathcal{R} \approx 0$

pick a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Note: $\vec{\Phi}(R_{ij}) \;\approx\;$ the parallelogram determined by $\Delta u\overrightarrow{T_{u_i}}$ and $\Delta v\overrightarrow{T_{v_j}}$.

Then
$$\begin{aligned}
\textstyle\iint_{\mathcal{S}} f\,dS \;&=\; \textstyle\iint_D f\left(\vec{\Phi}\,(u,v)\right)\;\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\;du\,dv\\
&\approx\; \textstyle\sum_{R_{ij}\in\mathcal{R}} \left[\,f(\vec{\Phi}(c_{ij}))\;\left\|\overrightarrow{T_{u_i}} \times \overrightarrow{T_{v_j}}\right\|\,\right]\;\Delta u\,\Delta v\\
&=\; \textstyle\sum_{R_{ij}\in\mathcal{R}} f(\vec{\Phi}(c_{ij}))\;\left[\,\left\|\overrightarrow{T_{u_i}} \times \overrightarrow{T_{v_j}}\right\|\;\Delta u\,\Delta v\,\right]\\
&=\; \textstyle\sum_{R_{ij}\in\mathcal{R}} f(\vec{\Phi}(c_{ij}))\;\left[\,\left\|\Delta u\,\overrightarrow{T_{u_i}} \times \Delta v\,\overrightarrow{T_{v_j}}\right\|\,\right]\\
&\approx\; \textstyle\sum_{R_{ij}\in\mathcal{R}} f(\vec{\Phi}(c_{ij}))\;\;\text{Area}\;(\vec{\Phi}(R_{ij}))\ .
\end{aligned}$$

7.e.3. Examples: compare with **7.b.3**

▷ $\iint_{\mathcal{S}} 1\,dS \;=\;$ surface area of $\mathcal{S}$.

▷ the AVERAGE VALUE of f on $\mathcal{S}$ is $\frac{\iint_{\mathcal{S}} f\,dS}{\iint_{\mathcal{S}} f\,dS}$

e.g. $f(x,y,z)$ is the temperature at the point (x,y,z)

so average value of f on $\mathcal{S}$ $=$ average temperature of f on $\mathcal{S}$.

▷ If $\mathcal{S}$ represents a surface in $\mathbb{R}^3$

with $\rho\ \mathbb{R}^3 \to \mathbb{R}$ the mass density (*e.g.*, grams/cm^2) of $\mathcal{S}$

then the total mass of the surface $\mathcal{S}$ $\;=\;$ $\iint_{\mathcal{S}} \rho\,dS$

and the center of mass of the surface $\mathcal{S}$ $\;=\;$ $\left(\frac{\iint_{\mathcal{S}} x\rho\,dS}{\iint_{\mathcal{S}} \rho\,dS}\ ,\ \frac{\iint_{\mathcal{S}} y\rho\,dS}{\iint_{\mathcal{S}} \rho\,dS}\ ,\ \frac{\iint_{\mathcal{S}} z\rho\,dS}{\iint_{\mathcal{S}} \rho\,dS}\right)$.

7.e.4. The integral of f over $\mathcal{S}$ is independent of the parameterization $\vec{\Phi}$,

i.e., using a different (but still NICE) parameterization of $\mathcal{S}$ yields the same number.

7.f. Oriented Surfaces

7.f.1. Consider a parametrized surface $\vec{\Phi}\colon \underset{\mathbb{R}^2}{\overset{\cap}{D^2}} \xrightarrow{\text{onto}} \underset{\mathbb{R}^3}{\overset{\cap}{\mathcal{S}}}$

Intuitively, $\mathcal{S}$ is an ORIENTED SURFACE if it has 2 sides:

▷ one side is the outside, or POSITIVE SIDE

▷ the other side is the inside, or NEGATIVE SIDE .

At each point $\vec{\Phi}(u,v) \in \mathcal{S}$, there is a (unique) OUTWARD unit normal vector $\vec{n}_{uv}$ with

▷ $\vec{n}_{uv}$ pointing away from the positive side of $\mathcal{S}$

▷ $-\vec{n}_{uv}$ pointing away from the negative side of $\mathcal{S}$.

We say that $\{\vec{n}_{uv}\}_{(u,v)\in D^2}$ orient $\mathcal{S}$.

If $\mathcal{S}$ is smooth, then either

▷ $\dfrac{\overrightarrow{T_u} \times \overrightarrow{T_v}}{\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|}\Big|_{(u_0,v_0)} = \vec{n}_{u_0\,v_0}$ for all $(u_0,v_0) \in D^2$, so $\vec{\Phi}$ is ORIENTATION-PRESERVING

▷ $\dfrac{\overrightarrow{T_u} \times \overrightarrow{T_v}}{\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|}\Big|_{(u_0,v_0)} = -\vec{n}_{u_0\,v_0}$ for all $(u_0,v_0) \in D^2$, so $\vec{\Phi}$ is ORIENTATION-REVERSING

7.f.2. Example 1: Sphere

$\vec{\Phi}(\theta,\phi) \;=\; (\,\rho\sin\phi\cos\theta\,,\ \rho\sin\phi\sin\theta\,,\ \rho\cos\phi\,)$

$D^2 \;=\; \{(\theta,\phi)\colon 0 \leqslant \theta \leqslant 2\pi\ ,\ 0 \leqslant \phi \leqslant \pi\}$

$\overrightarrow{T_\theta} \times \overrightarrow{T_\phi} \;=\; -\,\langle \rho^2\sin^2\phi\cos\theta\,,\ \rho^2\sin^2\phi\sin\theta\,,\ \rho^2\sin\phi\cos\phi\rangle \;=\; -\,(\rho\ \sin\phi)\,\vec{\Phi}(\theta,\phi)$

$\left\|\overrightarrow{T_\theta} \times \overrightarrow{T_\phi}\right\| \;=\; \rho^2\ \sin\phi$

$\vec{\Phi}$ is orientation-reversing

7.f.3. Example 2: Special Case: $z = g(x,y)$

$\vec{\Phi}(x,y) \;=\; (x,y,g(x,y))$ where $g \in C^1$

$$\vec{n}_{x_0\,y_0} \overset{\mathbf{2.11}}{=} \frac{\langle -\frac{\partial g}{\partial x}\big|_{(x_0,y_0)}\,,\ -\frac{\partial g}{\partial y}\big|_{(x_0,y_0)}\,,\ 1\rangle}{\left\|\langle -\frac{\partial g}{\partial x}\big|_{(x_0,y_0)}\,,\ -\frac{\partial g}{\partial y}\big|_{(x_0,y_0)}\,,\ 1\rangle\right\|} \;=\; \begin{array}{c}\text{the upward normal unit vector to } \mathcal{S}\\ \text{at the point } (x_0,y_0,g(x_0,y_0))\in\mathcal{S}\end{array}$$

$\overrightarrow{T_x} \;=\; \langle 1,0,\frac{\partial g}{\partial x}\rangle$

$\overrightarrow{T_y} \;=\; \langle 0,1,\frac{\partial g}{\partial y}\rangle$

$\overrightarrow{T_x} \times \overrightarrow{T_y} \;=\; \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle$

$\left\|\overrightarrow{T_x} \times \overrightarrow{T_y}\right\| \;=\; \frac{\|\vec{m}\|}{|\vec{m}\,\bullet\,\vec{k}|}$ where $\vec{m}$ is *any* normal vector to $\mathcal{S}$ at the point

$\vec{\Phi}$ is orientation-preserving

7.g. Surface Integrals of Vector–Valued Functions

7.g.1. Given $\underset{\mathbb{R}^2}{\overset{\cap}{D^2}} \xrightarrow[\text{onto}]{\vec{\Phi}} \underset{\mathbb{R}^3}{\overset{\cap}{\mathcal{S}}} \xrightarrow{\vec{F}} \mathbb{R}$

with $\vec{\Phi}$ is NICE and $\vec{F}$ is continuous

then the SURFACE INTEGRAL OF $\vec{F}$ OVER $\mathcal{S}$ is : $\boxed{\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} \;=\; \iint_{D^2} \vec{F} \bullet \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) \, du\, dv \;.}$

7.g.2. If $\vec{\Phi}$ is (any) orientation-preserving parmaterization of the smooth surface $\mathcal{S}$

then $\vec{n} \;=\; \dfrac{\left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right)}{\left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|}$

and so $\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} \;=\; \iint_{\mathcal{S}} \left(\vec{F} \bullet \vec{n}\right) dS$.

If $\begin{cases} \vec{\Phi}_1 \\ \vec{\Phi}_2 \end{cases}$ are $\begin{cases} \text{orientation-preserving} \\ \text{orientation-reversing} \end{cases}$ parmaterizations of the smooth surface $\mathcal{S}$

then $\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} \;=\; \iint_{\vec{\Phi}_1} \vec{F} \bullet d\vec{S} \;=\; -\iint_{\vec{\Phi}_2} \vec{F} \bullet d\vec{S}$.

7.g.3. Key Idea: using notation from **5.2** (see text pp. 431 & 455)

Form a partition $\mathcal{R} = \{R_{ij} \,:\, R_{ij} \subset D^2\}$ of D^2 into rectangles

pick a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Note: $\vec{\Phi}(R_{ij}) \;\approx\;$ the parallelogram determined by $\overrightarrow{T_{u_i}}\Delta u$ and $\overrightarrow{T_{v_j}}\Delta v$.

Then
$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} &= \iint_D \vec{F}\left(\vec{\Phi}(u,v)\right) \bullet \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right) du\, dv \\
&\approx \sum_{R_ij \in \mathcal{R}} \left(\vec{F}(\vec{\Phi}(c_{ij})) \bullet \left(\overrightarrow{T_{u_i}} \times \overrightarrow{T_{v_j}}\right)\right) \Delta u\, \Delta v \\
&= \sum_{R_ij \in \mathcal{R}} \vec{F}(\vec{\Phi}(c_{ij})) \bullet \left(\left(\overrightarrow{T_{u_i}} \times \overrightarrow{T_{v_j}}\right) \Delta u\, \Delta v\right) \\
&= \sum_{R_ij \in \mathcal{R}} \vec{F}(\vec{\Phi}(c_{ij})) \bullet \left(\overrightarrow{T_{u_i}}\, \Delta u \times \overrightarrow{T_{v_j}}\, \Delta v\right) .
\end{aligned}$$

▷ Let $P(\vec{A}, \vec{B}, \vec{C})$ be the parallelepiped determined by $\vec{A}$, $\vec{B}$, and $\vec{C}$,

$\vec{F}$ be the velocity field of a fluid ,

$\mathcal{S}$ be a porous surface through which the fluid flows ,

$\vec{\Phi}$ be a orientation-preserving parmaterization of the smooth surface $\mathcal{S}$

If $\vec{F}(\vec{\Phi}(c_{ij}))$ point in the $\begin{cases} \text{same} \\ \text{opposite} \end{cases}$ direction as $\overrightarrow{T_{u_i}} \times \overrightarrow{T_{v_j}}$, ie, in the $\begin{cases} \text{same} \\ \text{opposite} \end{cases}$ direction as $\vec{n}$

then
$$\begin{aligned}
\vec{F}(\vec{\Phi}(c_{ij})) \bullet \left(\overrightarrow{T_{u_i}\,\Delta u} \times \overrightarrow{T_{v_j}\,\Delta v}\right) &= (\pm) \text{ volume of } P\left(\vec{F}(\vec{\Phi}(c_{ij}))\,,\, \overrightarrow{T_{u_i}}\,\Delta u\,,\, \overrightarrow{T_{v_j}}\,\Delta v\right) \\
&= (\pm) \text{ amt. of fluid that flows } \begin{cases} \text{thru/out-of} \\ \text{thru/into} \end{cases} \mathcal{S} \text{ per unit time} .
\end{aligned}$$

So
$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} &= [(\text{amt. of fluid out-of } \mathcal{S}) - (\text{amt. of fluid into } \mathcal{S})] \text{ per unit time} \\
&= \text{the net quantity of fluid to flow across } \mathcal{S} \text{ per unit time} \\
&= \text{the rate of fluid flow} \\
&= \text{the FLUX of } \vec{F} \text{ across } \mathcal{S}.
\end{aligned}$$

7.h. Ch 7 – Summary

7.h.1. Given a path $[a,b] \xrightarrow{\vec{c}} \mathbb{R}^n$

a surface $\underset{\mathbb{R}^2}{\overset{D^2}{\cap}} \xrightarrow[\text{onto}]{\vec{\Phi}} \underset{\mathbb{R}^3}{\overset{\mathcal{S}}{\cap}}$

a scalar field $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

a vector field $\mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$

with everything nice & continuous.

7.b $\int_{\vec{c}} f\, ds = \int_a^b f(\vec{c}(t))\, \|\vec{c}\,'(t)\|\, dt$

$=$ averages f along the path $\vec{c}$

7.c $\int_{\vec{c}} \vec{F} \bullet d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \bullet \vec{c}\,'(t)\, dt$

$=$ work done by force $\vec{F}$ on a puffo as he moves along the path $\vec{c}$

7.e $\iint_{\mathcal{S}} f\, dS = \iint_{D^2} f\left(\vec{\Phi}\,(u,v)\right)\, \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\, du\, dv$

$=$ averages f over the surface $\mathcal{S}$

7.g $\iint_{\mathcal{S}} \vec{F} \bullet d\vec{S} = \iint_{D^2} \vec{F} \bullet \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right)\, du\, dv$

$= \iint_{\mathcal{S}} \left(\vec{F} \bullet \vec{n}\right)\, dS$

$=$ the flux of a vector field $\vec{F}$ across the surface $\mathcal{S}$

7.h.2. Integrals over paths

$\vec{c}(t) = \langle x(t), y(t), z(t)\rangle$

$\vec{c}\,'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle$

$ds = \|\vec{c}\,'\|\, dt$

$d\vec{s} = \vec{c}\,'\, dt$

7.h.3. Integrals over Surfaces

$\vec{\Phi}(u,v) = (\, x\,(u,v)\,, y\,(u,v)\,, z\,(u,v)\,)$

$\overrightarrow{T_u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle$

$\overrightarrow{T_v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$

$dS = \left\|\overrightarrow{T_u} \times \overrightarrow{T_v}\right\|\, du\, dv = \sqrt{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(u,v)}\right)^2}\, du\, dv$ where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\underset{\substack{\uparrow \\ \mathbf{7.g}}}{d\vec{S}} = \left(\overrightarrow{T_u} \times \overrightarrow{T_v}\right)\, du\, dv = \vec{n}\, \underset{\substack{\uparrow \\ \mathbf{7.e}}}{dS}$ where $\vec{n}$ is the outward unit normal vector to $\mathcal{S}$