Ch 6: Change of Variables

6.1. Recall the 1D Change of Variables Formula:

Have: $I^* \xrightarrow{g} I \xrightarrow{f} \mathbb{R}$

where: $I^* = [a,b] \subset \mathbb{R}$

$g \in C^1(I^*)$, i.e. g' is continuous on I^*

and (in general) g' is either positive or negative on I^*

$I = g(I^*) = [c,d]$ (with $c = \min_{[a,b]} f$ and $d = \max_{[a,b]} f$)

f is continuous .

Then:
$$\underbrace{\int_{g(a)}^{g(b)} f(x)\,dx}_{\text{want but hard to compute}} \overset{(*)}{=} \underbrace{\int_a^b f(g(u))\,\frac{dg}{du}\,du}_{\text{easy to compute}} .$$

usually:

If: $\frac{dg}{du} \geqslant 0$ on I^* (and so $I = [g(a), g(b)]$)

or: $\frac{dg}{du} \leqslant 0$ on I^* (and so $I = [g(b), g(a)]$)

Then: formula $(*)$ becomes $\int_I f(x)\,dx \overset{(**)}{=} \int_{I^*} f(g(u)) \left|\frac{dg}{du}\right| du$.

Goal the 2D & 3D Change of Variables Formulas:

Have: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$

$D^*, D \subset \mathbb{R}^{2 \text{ or } 3}$

$T(D^*) = D$

D^*, D, T, f are *nice* .

Want: an anolog to $(**)$,

like: $\iint_D f(x,y)\,dA \overset{(2D)}{=} \iint_{D^*} f(T(u,v)) \boxed{?}\, dA$

or: $\iiint_D f(x,y,z)\,dV \overset{(3D)}{=} \iiint_{D^*} f(T(u,v,w)) \boxed{?}\, dV$.

6.2. Interior & Boundary cf. **5.2** & **5.3**

For: the elementary region $R^2 = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b \text{ and } g_1(x) \leqslant y \leqslant g_2(x)\}$

the interior of R^2 is $\{(x,y) \in \mathbb{R}^2 : a < x < b \text{ and } g_1(x) < y < g_2(x)\}$

the boundary of R^2 is $R^2 \setminus [\text{the interior of } R^2]$.

For: the z-simple region $R^3 = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2 \text{ and } g_1(x,y) \leqslant z \leqslant g_2(x,y)\}$

the interior of R^3 is $\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in [\text{the interior of } R^2] \;\&\; g_1(x,y) < z < g_2(x,y)\}$

the boundary of R^3 is $R^3 \setminus [\text{the interior of } R^3]$.

6.3. One-to-One & Onto & Jacobian

Have: $T\colon D^* \to D$

$D^*, D \subset \mathbb{R}^n$

Def: T is one-to-one $\Leftrightarrow$ if $\vec{a}_i \in D^*$ and $\vec{a}_1 \neq \vec{a}_2$ then $T(\vec{a}_1) \neq T(\vec{a}_2)$

$\Leftrightarrow$ if $\vec{a}_i \in D^*$ and $T(\vec{a}_1) = T(\vec{a}_2)$ then $\vec{a}_1 = \vec{a}_2$

$\Leftrightarrow$ T does NOT send two different points of D^* to the same point in D

$\Leftrightarrow$ T does NOT fold-up D

$\overset{n=1}{\Leftrightarrow}$ T passes the hortizontal line test.

Rmk: in **6.1**, if g is 1-to-1, then either $\frac{dg}{du} \geqslant 0$ or $\frac{dg}{du} \leqslant 0$ on I^* .

Def: T is onto $\Leftrightarrow$ $T(D^*) = D$

$\Leftrightarrow$ if $\vec{b} \in D$ then there is $\vec{a} \in D^*$ so that $T(\vec{a}) = \vec{b}$

$\Leftrightarrow$ T maps D^* onto *all* of D .

If: $T \in C'(D^*)$

then: Jacobian matrix of T $= J_T = D_{_M} T$ (recall $D_{_M}$ was defined in **2.2**)

and: Jacobian of T $= \det J_T$.

So : for $n = 2$,

$$T(u,v) = \langle\, x(u,v)\,,\, y(u,v)\,\rangle$$
$$\det J_T = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and for $n = 3$,

$$T(u,v,w) = \langle\, x(u,v,w)\,,\, y(u,v,w)\,,\, z(u,v,w)\,\rangle$$
$$\det J_T = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} .$$

To change to cylindrical coordinates : (see **1.8**)

let: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

so: $T(r,\theta,z) = \langle r\cos\theta, r\sin\theta, z\rangle$

and: $\det J_T = r$ and $|\det J_T| = r$

To change to spherical coordinates : (see **1.9**)

let: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

so: $T(\rho,\theta,\phi) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$

and: $\det J_T = -\rho^2\sin\phi$ and $|\det J_T| = \rho^2\sin\phi$

6.4. <u>Examples</u> in $\mathbb{R}^2$: $\quad T\colon D^* \to D$

6.4.1. Polar Coordinates:

$\triangleright$. $T(r,\theta) \;=\; \langle r\cos\theta, r\sin\theta\rangle \quad\Rightarrow\quad J_T \;=\; \underline{\hspace{3cm}} \quad\Rightarrow\quad \det J_T \;=\; \underline{\hspace{2cm}}$

a. $D^* = [a,b]\times[0,\pi/2]$ where $0<a<b \quad\Rightarrow\quad T(D^*) = ?$

b. $D^* = [a,b]\times[0,3\pi]$ where $0<a<b \quad\Rightarrow\quad T(D^*) = ?$

6.4.2. Linear Transformations: here $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\triangleright$. $T(u,v) \;=\; \mathbf{A}\begin{bmatrix} u \\ v \end{bmatrix} + \vec{b} \;=\; \begin{bmatrix} a_{11}u + a_{12}v + b_1 \\ a_{21}u + a_{22}v + b_2 \end{bmatrix} \;=\; \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$

Facts from linear algebra:

(i) $\det \mathbf{A} \neq 0 \quad\Leftrightarrow\quad T$ is 1-to-1 (see § 6.1 Ex. 8)

(ii) $\det \mathbf{A} \neq 0 \quad\Leftrightarrow\quad T(\mathbb{R}^2) = \mathbb{R}^2$ (see § 6.1 Ex. 9)

(iii) $\mathbf{A}\vec{d}_1 \times \mathbf{A}\vec{d}_2 \;=\; (\det \mathbf{A})\,(\vec{d}_1 \times \vec{d}_2)$ (just write it out)

(iv) $J_T \;=\; \underline{\hspace{4cm}}$.

Consider parallelograms: here $\vec{d}_i \neq \vec{0}$

Let: $D^* = \{\vec{p} + \lambda\vec{d}_1 + \mu\vec{d}_2 : 0 \leqslant \lambda,\mu \leqslant 1\}$

so: D^* is a parallelogram $\Leftrightarrow \vec{d}_1 \nparallel \vec{d}_2 \Leftrightarrow \vec{d}_1 \times \vec{d}_2 \neq 0$.

Note: $T(D^*) = \{(\mathbf{A}\vec{p} + \vec{b}) + \lambda\,\mathbf{A}\vec{d}_1 + \mu\,\mathbf{A}\vec{d}_2 : 0 \leqslant \lambda,\mu \leqslant 1\}$.

So: $T(D^*)$ is a parallelogram $\Leftrightarrow$ D is a parallelogram and $\underline{\hspace{5cm}}$.

Note: $T([0,1]\times[0,1]) \;=\; \left\{ \vec{b} + \lambda\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + \mu\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} : 0 \leqslant \lambda,\mu \leqslant 1 \right\}$.

If: D^* & D are parallelograms, then

$$\text{area } T(D^*) \;=\; \left\| (\mathbf{A}\vec{d}_1) \times (\mathbf{A}\vec{d}_2) \right\| \overset{\text{(iii)}}{=} |\det \mathbf{A}|\, \left\| \vec{d}_1 \times \vec{d}_2 \right\| \;=\; |\det \mathbf{A}| \text{ area } (D) \overset{\text{(iv)}}{=} |\det J_T| \text{ area } (D) .$$

6.5. Change of Variables Formula

Let: $D^* \xrightarrow{T} D \xrightarrow{f} \mathbb{R}$

$D^*, D \subset \begin{bmatrix} \mathbb{R}^2 \\ \mathbb{R}^3 \end{bmatrix}$ and D^* is $\begin{bmatrix} \text{elementary} \\ \text{simple} \end{bmatrix}$ (see $\begin{bmatrix} \mathbf{5.2} \\ \mathbf{5.3} \end{bmatrix}$)

$T \in C'(D^*)$

T is 1-to-1 on the interior of D^*

$T(D^*) = D$

f is continuous .

For: 2D-case:

$$\iint_D f(x,y)\, dA_{(x,y)} = \iint_{D^*} f(T(u,v))\ |\det J_T|\ dA_{(u,v)}$$

mnemonically:

$$\iint_D f(x,y) \text{ “}dxdy\text{”} = \iint_{D^*} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \text{ “}dudv\text{” } .$$

For: 3D-case:

$$\iiint_D f(x,y,z)\, dV_{(x,y,z)} = \iiint_{D^*} f(T(u,v,w))\ |\det J_T|\ dV_{(u,v,w)}$$

mnemonically:

$$\iiint_D f(x,y,z) \text{ “}dxdydz\text{”} = \iiint_{D^*} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \text{ “}dudvdw\text{” } .$$