Math 550 – INFORMAL SUMMARY – Fall 1997

Ch 5: Double and Triple Integrals

5.1. Single Integral

Given: a function $f\colon [a,b] \to \mathbb{R}$

a partition $P\colon a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$

a selection $\{c_i\}_{i=1}^n$ where $c_i \in [x_{i-1}, x_i]$

Defs: $l\,(\,[x_{i-1}, x_i]\,) \;=\;$ length $[x_{i-1}, x_i]$

mesh of $P = |P| \;=\; \max_{1 \leqslant i \leqslant n} l\,(\,[x_{i-1}, x_i]\,)$

Reimann sum of $(f, P, \{c_i\}) = \sum_{i=1}^n\; f(c_i) \;\cdot\; l\,(\,[x_{i-1}, x_i]\,)$

DEF: f is integrable over $[a,b]$ with $\int_{[a,b]} f(x)\,dx \;=\; I \quad \stackrel{\text{def}}{\Longleftrightarrow}$

$\lim_{|P| \to 0} \sum_{i=1}^n\; f(c_i) \;\cdot\; l\,(\,[x_{i-1}, x_i]\,) \;=\; I$ for each selection of $\{c_i\}$'s .

So: $\int_{[a,b]} 1\; dx \;=\; l\,(\,[a,b]\,)$

and: $\sum_{i=1}^n\; f(c_i) \;\cdot\; \frac{l(\,[x_{i-1}, x_i]\,)}{l(\,[a,b]\,)} \;\approx\;$ avg. value of f over $[a,b]$ $\;=\; \dfrac{\int_{[a,b]} f(x)\,dx}{l\,(\,[a,b]\,)}$.

Def: the area of the region btw. the graph of $y = f(x)$ and $[a,b]$ $\stackrel{\text{def}}{=} \int_{[a,b]} f(x)\,dx$.

Thm: If f is continuous on $[a,b]$, then f is integrable over $[a,b]$.

Fundamental Theorem of Calculus: If $f = F'$ is continuous, then $\int_{[a,b]} f(x)\,dx \;=\; F(b) - F(a)$.

5.2. Double Integral

Given: $R^2 \subset [a_x, b_x] \times [a_y, b_y] \subset \mathbb{R}^2$

a function $f\colon R^2 \to \mathbb{R}$

a partition $P_x\colon a_x = x_0 < x_1 < \ldots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y\colon a_y = y_0 < y_1 < \ldots < y_m = b_y$ of $[a_y, b_y]$.

Form: rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Pick: a selection $\{c_{ij}\}$ where $c_{ij} \in R_{ij}$.

Defs: $A\,(R_{ij}) \;=\;$ area of R_{ij}, $\qquad \mathcal{R} = \{R_{ij} \;:\; R_{ij} \subset R^2\}$

mesh of $\mathcal{R} = |\mathcal{R}| \;=\; \max_{ij} A\,(R_{ij})$

Reimann sum of $(f, P_x, P_y, \{c_{ij}\}) = \sum_{R_{ij} \in \mathcal{R}}\; f(c_{ij}) \;\cdot\; A\,(R_{ij})$.

DEF: f is integrable over R^2 with $\iint_{R^2} f(x,y)\; dA \;=\; I \quad \stackrel{\text{def}}{\Longleftrightarrow}$

$\lim_{|\mathcal{R}| \to 0} \sum_{R_{ij} \in \mathcal{R}}\; f(c_{ij}) \;\cdot\; A\,(R_{ij}) \;=\; I$ for each selection of $\{c_{ij}\}$'s .

So: $\iint_{R^2} 1\; dA \;=\;$ area of R^2

and: $\sum_{R_{ij} \in \mathcal{R}}\; f(c_{ij}) \;\cdot\; \frac{A(R_{ij})}{A(R^2)} \;\approx\;$ avg. value of f over R^2 $\;=\; \frac{\iint_{R^2} f(x,y)\; dA}{\text{area of } R^2}$.

Def: the volume of the solid btw. the graph of $z = f(x, y)$ and $R^2 \stackrel{\text{def}}{=} \iint_{R^2} f(x, y)\, dA$.

Def: R^2 is elementary if, for continuous functions g_i, either:

R^2 is of $\underline{\text{type } 1 = \text{type}_y}$ $\stackrel{\text{def}}{\Longleftrightarrow}$ $R^2 = \{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b \text{ and } g_1(x) \leqslant y \leqslant g_2(x)\}$

$\leftrightsquigarrow$ draw typical rectangle $\|$ to y-axis

R^2 is of $\underline{\text{type } 2 = \text{type}_x}$ $\stackrel{\text{def}}{\Longleftrightarrow}$ $R^2 = \{(x, y) \in \mathbb{R}^2 : g_1(y) \leqslant x \leqslant g_2(y) \text{ and } a \leqslant y \leqslant b\}$

$\leftrightsquigarrow$ draw typical rectangle $\|$ to x-axis

R^2 is of $\underline{\text{type } 3 = \text{type}_{xy}}$ $\stackrel{\text{def}}{\Longleftrightarrow}$ R^2 is of both type_x and type_y .

If: $f \colon R^2 \to \mathbb{R}$ is continuous and R^2 is elementary,

then: $\iint_{R^2} f(x, y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\right] dx$ if R^2 is of type_y

and: $\iint_{R^2} f(x, y)\, dA = \int_a^b \left[\int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\right] dy$ if R^2 is of type_x .

5.3. Triple Integral

Given: $R^3 \subset [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \subset \mathbb{R}^3$

a function $f \colon R^3 \to \mathbb{R}$

a partition $P_x \colon a_x = x_0 < x_1 < \ldots < x_n = b_x$ of $[a_x, b_x]$

a partition $P_y \colon a_y = y_0 < y_1 < \ldots < y_m = b_y$ of $[a_y, b_y]$

a partition $P_z \colon a_z = z_0 < z_1 < \ldots < z_l = b_z$ of $[a_z, b_z]$.

Form: rectangular cubes (ie. boxes) $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$.

Pick: a selection $\{c_{ijk}\}$ where $c_{ijk} \in B_{ijk}$.

Defs: $V(B_{ijk}) =$ volume of B_{ijk}, $\mathcal{V} = \{B_{ijk} : B_{ijk} \subset R^3\}$

mesh of $\mathcal{V} = |\mathcal{V}| = \max_{ijk} V(B_{ijk})$

Reimann sum of $(f, P_x, P_y, P_z, \{c_{ijk}\}) = \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk})$.

DEF: f is integrable over R^3 with $\iiint_{R^3} f(x, y, z)\, dV = I$ $\stackrel{\text{def}}{\Longleftrightarrow}$

$\lim_{|\mathcal{V}| \to 0} \sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot V(B_{ijk}) = I$ for each selection of $\{c_{ijk}\}$'s .

So: $\iiint_{R^3} 1\, dV =$ volume of R^3

and: $\sum_{B_{ijk} \in \mathcal{V}} f(c_{ijk}) \cdot \frac{V(B_{ijk})}{V(R^3)} \approx$ avg. value of f over $R^3 = \frac{\iiint_{R^3} f(x,y,z)\, dV}{V(R^3)}$

Def: R^3 is z-simple if, $R^3 = \{(x, y, z) \in \mathbb{R}^3 \colon (x, y) \in \mathbb{R}^2 \text{ and } g_1(x, y) \leqslant z \leqslant g_2(x, y)\}$ where :

▷ $g_i \in C^1$, ie. g_i have continuous 1^{st}-order partial derivatives

▷ R^2 is elementary , as in **5.2** .

$\rightsquigarrow$ draw typical box with height $\|$ to z-axis

If: f is continuous and R^3 is z-simple,

then: $\iiint_{R^3} f(x, y, z)\, dV = \iint_{R^2} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\right] dA$.

Rmk: Analogs hold for x-simple and y-simple.