Math 550 – INFORMAL SUMMARY – Fall 1997

Ch 3: Higher-Order Derivatives

3.1. Defs./Theorem

Given: a scalar field $f\colon \mathbb{D}^n \to \mathbb{R}$ where $\mathbb{D}^n \subset \mathbb{R}^n$

so f has the form $f(\vec{x}) = w$, or equiv. $f(x_1, \ldots, x_n) = w$

Then: an iterated partial derivative of f of order k is $\frac{\partial^k f}{\partial x_{i_k} \cdots \partial x_{i_2} \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}}\left(\cdots\left(\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f}{\partial x_{i_1}}\right)\right)\right)$

f is of class C^k if all the iterated partial derivative of f of order k are continuous.

Thm: If f is of class C^2 , then $\frac{\partial^2 f}{\partial x_{i_2} \partial x_{i_1}} = \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}$.

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§4.3–4.4 Vector Fields and Div and Curl

4.1. Vector Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$, a function $\vec{F}\colon \mathbb{D}^n \to \mathbb{R}^n$ *i.e.*, $F_i\colon \mathbb{D}^n \to \mathbb{R}$

so $\vec{F}$ has the form $\vec{F}(x_1, \ldots, x_n) = \langle F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)\rangle$

Then: $\vec{F}$ is called a vector field and F_i are called the component scalar fields

$\vec{F}$ is of class C^k if each F_i is of class C^k .

Given: a path $\vec{c}\colon \mathbb{D}^1 \to \mathbb{R}^n$

so $\vec{c}$ has the form $\vec{c}(t) = \langle x_1(t), \ldots, x_n(t)\rangle$ where $x_i\colon \mathbb{D}^1 \to \mathbb{R}$.

If: $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$

Then: $\vec{c}$ is called a flow line of $\vec{F}$

$\vec{c}$ is the sol'n to the system of differential eqs: $x_i'(t) = F_i(x_1(t), \ldots, x_n(t))$

$\vec{F}(\vec{c}(t_0))$ is tangent to the curve $\vec{c}$ at the point $\vec{c}(t_0)$

$n = 3$: if $\vec{F}$ is the velocity field of a fluid, then $\vec{c}$ is the path followed by a particle in the fluid .

4.2. Divergence & Curl Let $\begin{bmatrix}\mathcal{S}\\ \mathcal{V}\end{bmatrix}$ be the class of all the (nice) $\begin{bmatrix}\text{scalar}\\ \text{vector}\end{bmatrix}$ fields.

Given: a vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ where $\mathbb{D}^3 \subset \mathbb{R}^3$

the vector differential operator $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \bullet \nabla$

Defs:

$$\overrightarrow{\text{grad}}\, f = \nabla(f) \quad ; \quad \text{div } \vec{F} = \nabla \bullet \vec{F} \quad ; \quad \text{curl } \vec{F} = \nabla \times \vec{F} .$$

So:

$$\mathcal{S} \xrightarrow{\overrightarrow{\text{grad}}} \mathcal{V} \xrightarrow{\text{curl}} \mathcal{V} \xrightarrow{\text{div}} \mathcal{S}$$

$$\mathcal{S} \xrightarrow{\nabla^2} \mathcal{S} \quad \text{and} \quad \mathcal{V} \xrightarrow{\nabla^2} \mathcal{V}$$

Given: a function $\vec{F} \colon \mathbb{D}^2 \to \mathbb{R}^2$

so $\vec{F}$ has the form $\vec{F} = \langle F_1, F_2 \rangle$

Then: <u>scalar curl</u> of $\vec{F} \; = \; \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

4.3. <u>Basic Identities</u>

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$ where $c \in R$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ at points $\vec{x}$ where $g(\vec{x}) \neq 0$
5. $\text{div } (\vec{F} + \vec{G}) = \text{div } \vec{F} + \text{div } \vec{G}$
6. $\text{curl } (\vec{F} + \vec{G}) = \text{curl } \vec{F} + \text{curl } \vec{G}$
7. $\text{div } (f\vec{F}) = f \,\text{div } \vec{F} + \vec{F} \bullet \nabla f$
8. $\text{div } (\vec{F} \times \vec{G}) = \vec{G} \bullet \text{ curl } \vec{F} - \vec{F} \bullet \text{ curl } \vec{G}$
9. $\text{div curl } \vec{F} = 0$ if $\vec{F} \in C^2$
10. $\text{curl } (f\vec{F}) = f\text{curl } \vec{F} + \nabla f \times \vec{F}$
11. $\text{curl } \nabla f = \vec{0}$ if $f \in C^2$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \bullet \nabla g)$
13. $\text{div } (\nabla f \times \nabla g) = 0$ if $f, g \in C^2$
14. $\text{div } (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

4.4. <u>Gradient Vector Field</u>

Given: a vector field $\vec{F} \colon \mathbb{D}^3 \to \mathbb{R}^3$ where $\mathbb{D}^3 \subset \mathbb{R}^3$

a scalar field $f \colon \mathbb{D}^3 \to \mathbb{R}$ and $f \in C^1$

$\nabla f = \vec{F}$

Then: $\vec{F}$ is called a <u>gradient vector field</u>

f is called the <u>potential</u> of $\vec{F}$.

<u>Thm</u>. If: $\vec{F} \colon \mathbb{D}^3 \to \mathbb{R}^3$ is a vector field

$\vec{F} \in C^1$

curl $F \neq \vec{0}$

Then: $\vec{F}$ is NOT a gradient vector field .

Why: by 4.3 #11 above .

4.5. <u>Geometry of the Divergence</u> pg 284-5

Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a fluid

a point $(x_0, y_0, z_0) \in \mathbb{D}^3$ and a small $\varepsilon > 0$

$P(0)$ is the cube based at (x_0, y_0, z_0) with sides $\varepsilon\vec{i}, \varepsilon\vec{j}, \varepsilon\vec{k}$

$P(t)$ is (approx.) the parallelepiped that $P(0)$ is carried to t time-units later

$\mathcal{V}(t) =$ volume of $P(t)$.

Then: div $\vec{F}(x_0, y_0, z_0) = \frac{1}{\mathcal{V}(0)} \frac{d\mathcal{V}}{dt}\Big|_{\left[\substack{t=0 \\ (x,y,z)=(x_0,y_0,z_0)}\right]}$.

So: div $\vec{F}$ is the rate of expansion per unit volume under the flow of the fluid

div $\vec{F}(x_0, y_0, z_0) > 0 \;\Rightarrow\;$ the fluid is exanding at (x_0, y_0, z_0)

div $\vec{F}(x_0, y_0, z_0) < 0 \;\Rightarrow\;$ the fluid is compressing at (x_0, y_0, z_0) .

4.6. <u>Rotation</u> pg 278-9

Given: a (slanted/flat) solid rigid body B rotating about an axis L

some point Q on B

$\vec{v} =$ velocity vector field of B

$\vec{w} =$ angular velocity vector of B .

So:
- ▷ $\vec{w} \parallel L$
- ▷ $\|\vec{w}\| = \omega =$ angular speed of $B = \frac{\text{speed of } Q}{d(Q,L)}$.

Then: $\vec{v} = \vec{w} \times \overrightarrow{OQ}$

curl $\vec{v} = 2\vec{w}$.

4.7. <u>Geometry of the Curl</u> pg 279

Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a fluid

a point $(x, y, z) \in \mathbb{D}^3$

a RIGID body B centered at (x, y, z) .

Then: curl $\vec{F}(x, y, z)$ is twice the angular velocity vector of B

as it rotates as the fluid moves near (x, y, z) .

Def: curl $\vec{F} = 0 \overset{\text{def}}{\iff} \vec{F}$ is IRROTATIONAL .

4.8. <u>Key Idea</u>

Given: a VELOCITY vector field $\vec{F}\colon \mathbb{D}^3 \to \mathbb{R}^3$ of a fluid .

Then: div $\vec{F}$ measures how much the fluid is expanding

curl $\vec{F}$ measures how much the fluid is rotating .