$\boxed{\text{Math 550 – INFORMAL SUMMARY – Fall 1997}}$

$\boxed{\text{Ch 2: Differentiation} \quad \& \quad \S 4.1\text{–}4.2 \text{ Motion}}$

2.1. Defs.

a. the neighborhood of $x \in \mathbb{R}^n$ of radius r is $\{y \in \mathbb{R}^n : \|x - y\| < r\}$

b. open set - pg 94

c. limit - pg 98

d. continuous function - pg 104

2.2. Differentiation Rules

Given: an open subset $U^n \subset \mathbb{R}^n$ and $\vec{x}_0 \in U^n$

functions $\vec{f} \colon U^n \to \mathbb{R}^m$ and $\vec{g} \colon U^n \to \mathbb{R}^m$ that are each differentiable at $\vec{x}_0$

a constant $c \in \mathbb{R}$.

Then:

$$\nabla \left[c\, \vec{f}\right](\vec{x}_0) = c\, \nabla \left[c\, \vec{f}(\vec{x}_0)\right]$$

$$\nabla \left[\vec{f} + \vec{g}\right](\vec{x}_0) = \nabla \left[\vec{f}(\vec{x}_0)\right] + \nabla \left[\vec{g}(\vec{x}_0)\right]$$

Given: open subsets $U^n \subset \mathbb{R}^n$ and $V^m \subset \mathbb{R}^m$ and $\vec{x}_0 \in U^n$

functions $\vec{g} \colon U^n \to \mathbb{R}^m$ and $\vec{f} \colon V^m \to \mathbb{R}^p$

$\vec{g}(U^n) \subset V^m$ and so it makes sense to have $\vec{f} \circ \vec{g} \colon U^n \to \mathbb{R}^p$.

$\vec{g}$ is differentiable at $\vec{x}_0$ and $\vec{f}$ is differentiable at $\vec{g}(\vec{x}_0)$.

Then the Chain Rule says that :

$$\nabla \left(\vec{f} \circ \vec{g}\right)(\vec{x}_0) = \nabla \vec{f}\left(\vec{g}\left(\vec{x}_0\right)\right) \otimes \nabla \vec{g}(\vec{x}_0) .$$

2.3. Vector-Valued Functions

Given: an subset $\mathbb{D}^1 \subset \mathbb{R}^1$ and $t_0 \in \mathbb{D}^1$

a function $\vec{F} \colon \mathbb{D}^1 \to \mathbb{R}^m$

so $\vec{F}$ has the form $\boxed{\vec{F}(t) = \langle f_1(t), \ldots, f_m(t)\rangle}$ where $f_i \colon \mathbb{D}^1 \to \mathbb{R}$.

$\vec{F}$ is called a VECTOR-VALUED FUNCTION .

One thinks of $\vec{F}$ as tracing out (ie. parameterizing) a (1-D) space curve $\mathcal{C}$ in $\mathbb{R}^m$.

When $\mathbb{D}^1$ is an open set:

$$\frac{d\vec{F}}{dt} = \langle \frac{df_1}{dt}, \ldots, \frac{df_m}{dt} \rangle$$

2.4. Paths/Curves

Def. A PATH is a vector-valued function $\vec{c}\colon [a,b] \to \mathbb{R}^m$.

The CURVE or ARC $\mathcal{C}$ that is PARAMETERIZED by $\vec{c}$ is $\{\vec{c}(t) \in \mathbb{R}^m : a \leqslant t \leqslant b\}$.

Def. $\mathcal{C}$ is SMOOTH if there is a parametrization $\vec{F} = \vec{F}(t)$, $a \leqslant t \leqslant b$, such that:

i) $\frac{d\vec{F}}{dt}$ exists & is continuous on $[a,b]$

ii) if $a < t_1 < t_2 < b$, then $\vec{F}(t_1) \neq \vec{F}(t_2)$

iii) if $a \leqslant t_0 \leqslant b$, then $\frac{d\vec{F}(t_0)}{dt} \neq \vec{0}$.

Def. $\mathcal{C}$ is REGULAR if it consists of a finite number of smooth curves joined together and it does not cross itself. $\mathcal{C}$ is CLOSED if $\vec{c}(a) = \vec{c}(b)$.

Rmk. The UNIT TANGENT VECTOR to $\mathcal{C}$ at t_0 is $\vec{T} = \frac{\vec{c}\,'(t_0)}{\|\vec{c}\,'(t_0)\|}$.

Rmk. The ARCLENGTH of $\mathcal{C}$ is: $s(a,b) = \int_a^b \left\| \frac{d\vec{c}}{dt} \right\| \, dt$.

2.5. Differentiation Rules for vector-valued functions

Given: an open subset $\mathbb{D}^1 \subset \mathbb{R}^1$

differentiable functions $\vec{F}, \vec{G}\colon \mathbb{D}^1 \to \mathbb{R}^m$ and $p\colon \mathbb{D}^1 \to \mathbb{R}$

Then:

$$\frac{d}{dt}\Big[p(t)\,\vec{F}(t)\Big] = p'(t)\vec{F}(t) + p(t)\vec{F}'(t)$$

$$\frac{d}{dt}\Big[\vec{F}(t) \bullet \vec{G}(t)\Big] = \vec{F}'(t) \bullet \vec{G}(t) + \vec{F}(t) \bullet \vec{G}'(t)$$

$$\frac{d}{dt}\Big[\vec{F}(t) \times \vec{G}(t)\Big] = \vec{F}'(t) \times \vec{G}(t) + \vec{F}(t) \times \vec{G}'(t) .$$

Chain Rule: $\frac{d}{dt}\Big[\vec{F}(p(t))\Big] = p'(t)\vec{F}'(p(t))$.

2.6. Motion For a particle moving through space on a "nice" curve $\mathcal{C}$ with :

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ position vector

$\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt}$ velocity vector

$\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt}$ acceleration vector

$v(t) = \|\vec{v}(t)\| \neq \dfrac{d\,\|\vec{r}(t)\|}{dt}$ speed function

$\vec{T}(t) = \dfrac{\vec{v}(t)}{\|\vec{v}(t)\|}$ unit tangent vector to $\mathcal{C}$ at the pt. $\vec{r}(t)$

$\vec{N}(t) = \left\| \dfrac{d\vec{T}(t)}{dt} \right\|^{-1} \dfrac{d\vec{T}(t)}{dt}$ unit principle normal vector to $\mathcal{C}$ at the pt. $\vec{r}(t)$

$\vec{T}(t) \perp \vec{N}(t)$ with $\vec{T}(t) \parallel \vec{v}(t)$ & $\vec{N}(t) \perp \vec{v}(t)$

$\vec{a}(t) = a_T(t)\,\vec{T}(t) + a_N(t)\,\vec{N}(t)$ with $a_T(t) = \dfrac{dv(t)}{dt}$ & $a_N(t) = v(t)\left\| \dfrac{d\vec{T}(t)}{dt} \right\|$

$\|\vec{a}(t)\|^2 = |a_T(t)|^2 + |a_N(t)|^2$ & $a_N \geqslant 0$

$\vec{F} = m\vec{a}$ if a force $\vec{F}$ is acting on the particle of mass m

2.7. Scalar Fields

Given: a subset $\mathbb{D}^n \subset \mathbb{R}^n$ and a function $f\colon \mathbb{D}^n \to \mathbb{R}$

so f has the form $f(\vec{x}) = w$, or equiv. $f(x_1, \dots, x_n) = w$

Then: f is called a SCALAR FIELD

the GRAPH of $f = \left\{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} \colon (x_1, \dots, x_n,) \in \mathbb{D}^n\right\}$

the LEVEL SET of value $c \in \mathbb{R}$ for f is $\{\vec{x} \in \mathbb{D}^n : f(\vec{x}) = c\} \subset \mathbb{R}^n$

▷ $[n = 2] \Rightarrow$ [level set = LEVEL CURVE]

▷ $[n = 3] \Rightarrow$ [level set = LEVEL SURFACE]

the GRADIENT of $f = \nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$.

2.8. Curves on Surfaces

★ Given: an open set $\mathbb{D}^n \subset \mathbb{R}^n$ and a scalar field $f : \mathbb{D}^n \to \mathbb{R}$

$f(x_1, \dots, x_n) = w$

an interval $I \subset \mathbb{R}$ and a path $\vec{r}\colon I \to \mathbb{R}^n$

$\vec{r}(t) = \langle r_1(t), \dots, r_n(t) \rangle$

$\vec{r}(I) \subset \mathbb{D}^n$ so $F \stackrel{\text{def}}{=} f \circ \vec{r}\colon I \to \mathbb{R}$

$F(t) = f(r_1(t), \dots, r_n(t))$

When $n = 2$, think of F as a curve on the surface of f.

If f and $\vec{r}$ are *nice*, *i.e.*, differentiable, then

$$\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle \qquad \text{and} \qquad \frac{d\vec{r}}{dt} = \langle \frac{dr_1}{dt}, \dots, \frac{dr_n}{dt} \rangle$$

and the Chain Rule (2.4) gives:

$$\frac{dF}{dt}\Big|_{t_0} = \nabla f\,(\vec{r}\,(t_0)) \bullet \frac{d\vec{r}}{dt}\,(t_0) = \nabla f\big|_{\vec{r}(t_0)} \bullet \vec{r}\,'(t_0) \ .$$

► In ★, consider the level set $\mathcal{S}$ given by $f(\vec{x}) = k$:

If: $\vec{r}(I) \subset \mathcal{S}$ so $(f \circ \vec{r})\,(t) = k$ for each $t \in I$

$\vec{x}_0 \in \mathcal{S}$ so $f(\vec{x}_0) = k$

$t_0 \in I$ with $\vec{r}(t_0) = \vec{x}_0$

Then: $\nabla f\big|_{\vec{x}_0} \bullet \vec{r}\,'(t_0) = \frac{d}{dt}\,(f \circ \vec{r})\,\big|_{t_0} = 0$

so $\nabla f\big|_{\vec{x}_0} \perp$ (the tangent vector to $\vec{r}(t)$) $\big|_{t_0}$

so $\nabla f\big|_{\vec{x}_0} \perp \mathcal{S}\big|_{\vec{x}_0}$.

► In ★, consider the case where $\vec{r}$ is a line segment through $\vec{x}_0$ in the direction of $\vec{u}$:

Given: $\vec{x}_0 \in \mathbb{D}^n$ and $\vec{u} \in \mathbb{R}^n$ and so $\vec{r}(t) = \vec{x}_0 + t\vec{u}$.

The DIRECTIONAL DERIVATIVE of f at $\vec{x}_0$ in the direction of $\vec{u}$ is :

$$D_{\vec{u}} f(\vec{x}_0) \;=\; \frac{d(f \circ \vec{r})}{dt}\Big|_{\substack{\vec{x}=\vec{x}_0 \\ t=0}} \;=\; \lim_{t \to 0} \frac{f(\vec{x}_0 + t\vec{u}) - f(\vec{x}_0)}{t} \;=\; \vec{u} \bullet \nabla f\big|_{\vec{x}_0} \;=\; \|\vec{u}\| \; \Big\|\nabla f\big|_{\vec{x}_0}\Big\| \; \cos \measuredangle \vec{u} \nabla f\big|_{\vec{x}_0}$$

Thus $D_{\vec{u}} f\big|_{\vec{x}_0}$ is just the rate of change of f at $\vec{x}_0$ in the direction of $\vec{u}$ when $\|\vec{u}\| = 1$.

$\pm\, \nabla f$ points in the direction of the maximum rate of $\left[\begin{smallmatrix}\text{increase}\\ \text{decrease}\end{smallmatrix}\right]$ of f .

$\pm\, \|\nabla f\|$ equals the maximum rate of $\left[\begin{smallmatrix}\text{increase}\\ \text{decrease}\end{smallmatrix}\right]$ of f per unit distance .

2.9. Tangent Planes

Here: open sets $\mathbb{D}^n \subset \mathbb{R}^n$

▶ Given a scalar field $f \colon \mathbb{D}^3 \to \mathbb{R}$

$\vec{x}_0 = \langle x_0, y_0, z_0 \rangle \in \mathbb{D}^3$ with $f(\vec{x}_0) = k$

the level surface $\mathcal{S}$ given by $f(\vec{x}) = k$

$\nabla f\big|_{\vec{x}_0} \neq 0$.

Then: $\nabla f\big|_{\vec{x}_0} \perp \mathcal{S}\big|_{\vec{x}_0}$

the equation of the tangent plane to $\mathcal{S}$ at the point (x_0, y_0, z_0) is :

$$\nabla f\big|_{\vec{x}_0} \bullet \langle x - x_0, y - y_0, z - z_0 \rangle \;=\; 0 \,.$$

▶ Given: a scalar field $g \colon \mathbb{D}^2 \to \mathbb{R}$

Let: $f(x, y, z) = g(x, y) - z$

So: [the level surface $f(x, y, z) = 0$] $=$ [the surface $z = g(x, y)$] $\stackrel{\text{def}}{=}$ $\mathcal{S}$

So: $\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle\big|_{(x_0, y_0)} \perp \mathcal{S}\big|_{(x_0, y_0, g(x_0, y_0))}$

the equation of the tangent plane to $\mathcal{S}$ at the point $(x_0, y_0, g(x_0, y_0))$ is both :

$$\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \rangle\big|_{(x_0, y_0)} \bullet \langle x - x_0, y - y_0, z - g(x_0, y_0) \rangle \;=\; 0$$

$$z = g(x_0, y_0) \;+\; \frac{\partial g}{\partial x}\Big|_{(x_0, y_0)} (x - x_0) \;+\; \frac{\partial g}{\partial y}\Big|_{(x_0, y_0)} (y - y_0) \,.$$

▶ Given: a scalar field $h \colon \mathbb{D}^1 \to \mathbb{R}$

Let: $f(x, y, z) = h(x) - y$

So: [the level surface $f(x, y, z) = 0$ intersected with the plane $z = 0$]

$=$ [the graph of $y = h(x), z = 0$] $\stackrel{\text{def}}{=}$ $\mathcal{G}$

So: $\langle \frac{dh}{dx}, -1, 0 \rangle\big|_{(x_0)} \perp \mathcal{G}\big|_{(x_0, h(x_0), 0)}$

the equation of the tangent line to $\mathcal{G}$ at the point $(x_0, h(x_0), 0)$ is both :

$$\langle \frac{dh}{dx}, -1, 0 \rangle\big|_{(x_0)} \bullet \langle x - x_0, y - h(x_0), 0 \rangle \;=\; 0$$

$$y = h(x_0) \;+\; \frac{dh}{dx}\Big|_{(x_0)} (x - x_0) \quad \text{and} \quad z = 0 \,.$$