Math 550 – INFORMAL SUMMARY – Fall 97

Ch 1: Geometry

1.1. Vector - detererminded by its magnitude (i.e. length) and direction.

Remark: a POINT $P = (a, b, c)$ determines a VECTOR $\overrightarrow{OP} = \langle a, b, c \rangle$

1.2. Products Given: $\vec{A} = \langle a_1, a_2, a_3 \rangle$ and $\vec{B} = \langle b_1, b_2, b_3 \rangle$

$$0 \leqslant \measuredangle \vec{A}\vec{B} \stackrel{\text{def}}{=} \text{angle between } \vec{A} \;\&\; \vec{B} \stackrel{\text{def}}{=} \theta_{AB} \leqslant \pi$$

$$\vec{n}_{AB} \stackrel{\text{def}}{=} \text{the right-hand-rule unit vector} \perp \text{to } \vec{A} \text{ and } \vec{B}$$

direction angles $0 \leqslant \alpha, \beta, \gamma \leqslant \pi$ for $\vec{A}$.

Then:

$$\text{length of } \vec{A} = \left\|\vec{A}\right\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{A} \bullet \vec{A}}$$

$$\text{dot product} = \vec{A} \bullet \vec{B} = \left\|\vec{A}\right\| \left\|\vec{B}\right\| \cos\theta_{AB} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\text{cross product} = \vec{A} \times \vec{B} = \left[\left\|\vec{A}\right\| \left\|\vec{B}\right\| \sin\theta_{AB}\right] \vec{n}_{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\text{triple product} = \left[\vec{A}, \vec{B}, \vec{C}\right] = \vec{A} \times \vec{B} \bullet \vec{C} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{angle btw.} = \measuredangle \vec{A}\vec{B} = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{\left\|\vec{A}\right\| \left\|\vec{B}\right\|}\right)$$

$$\alpha = \measuredangle \vec{A}\,\vec{i} = \cos^{-1}\left(a_1 \,/\, \left\|\vec{A}\right\|\right)$$

$$\beta = \measuredangle \vec{A}\,\vec{j} = \cos^{-1}\left(a_2 \,/\, \left\|\vec{A}\right\|\right)$$

$$\gamma = \measuredangle \vec{A}\,\vec{k} = \cos^{-1}\left(a_3 \,/\, \left\|\vec{A}\right\|\right) \qquad \text{so } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

1.3. 1.2 in action

$\vec{A}$ is a UNIT vector $\Leftrightarrow$ $\left\|\vec{A}\right\| = 1$

the NORMALIZED vector of $\vec{A}$ is $\frac{\vec{A}}{\|\vec{A}\|}$

$\vec{A}$ & $\vec{B}$ are ORTHONORMAL $\leftrightarrow$ $\vec{A} \perp \vec{B}$ and $\left\|\vec{A}\right\| = 1 = \left\|\vec{B}\right\|$

$\vec{A} \perp \vec{B}$ $\leftrightarrow$ $\vec{A} \bullet \vec{B} = 0$ $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are PERPENDICULAR $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are ORTHOGONAL

$\vec{A} \| \vec{B}$ $\leftrightarrow$ $\vec{A} \times \vec{B} = 0$ $\leftrightarrow$ $\vec{A}$ & $\vec{B}$ are PARELLEL

the area of the triangle determined by $\vec{A}$ and $\vec{B}$ is $Area = \frac{1}{2}\left\|\vec{A} \times \vec{B}\right\|$

the area of the parallelogram determined by $\vec{A}$ and $\vec{B}$ is $Area = \left\|\vec{A} \times \vec{B}\right\|$

the volume of the parallelepiped determined by $\vec{A}, \vec{B}, \vec{C}$ is $Vol = \left|\ \left[\vec{A}, \vec{B}, \vec{C}\right]\ \right|$

1.4. Algebraic Properties

$$\left(s\vec{A}\right) \bullet \vec{B} \;=\; s\left(\vec{A} \bullet \vec{B}\right) \;=\; \vec{A} \bullet \left(s\vec{B}\right) \qquad \leftrightsquigarrow \qquad \left(s\vec{A}\right) \times \vec{B} \;=\; s\left(\vec{A} \times \vec{B}\right) \;=\; \vec{A} \times \left(s\vec{B}\right)$$

$$\vec{A} \bullet \vec{B} \;=\; \vec{B} \bullet \vec{A} \qquad \leftrightsquigarrow \qquad \vec{A} \times \vec{B} \;=\; -\left(\vec{B} \times \vec{A}\right)$$

$$\left(\vec{A} + \vec{B}\right) \bullet \vec{C} \;=\; \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C} \qquad \leftrightsquigarrow \qquad \left(\vec{A} + \vec{B}\right) \times \vec{C} \;=\; \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$$

$$\vec{A} \bullet \left(\vec{B} + \vec{C}\right) \;=\; \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \qquad \leftrightsquigarrow \qquad \vec{A} \times \left(\vec{B} + \vec{C}\right) \;=\; \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

1.5. Components

The parallel-perpendicular decomposition of $\vec{B}$ is $\vec{B} = \vec{B}_{\|} + \vec{B}_{\perp}$ where:

$$\vec{B}_{\|} = \frac{\vec{A} \bullet \vec{B}}{\vec{A} \bullet \vec{A}}\,\vec{A} \qquad \text{is } \| \text{ to } \vec{A}$$

$$\vec{B}_{\perp} = \frac{(\vec{A} \times \vec{B}) \times \vec{A}}{\vec{A} \bullet \vec{A}} \qquad \text{is } \perp \text{ to } \vec{A}\,.$$

$\vec{B}_{\|}$ is called the orthogonal projection of $\vec{B}$ onto $\vec{A}$.

The (signed) component of $\vec{B}$ along $\vec{A}$ is:

$$\begin{aligned} \operatorname{comp}_{\vec{A}}\vec{B} &\overset{\text{def}}{=} (\text{sign } \cos\theta_{AB})\,\left\|\vec{B}_{\|}\right\| \\ &\overset{1.2}{=} \left\|\vec{B}\right\|\cos\theta_{AB} \overset{1.2}{=} \frac{\vec{A} \bullet \vec{B}}{\left\|\vec{A}\right\|}\,. \end{aligned}$$

1.6. Equations and parameterizations

▷ The line $\mathcal{L}$ through $R_0 = (x_0,\ y_0,\ z_0)$ and parallel to $\vec{v}_0 = \langle a,\ b,\ c\rangle$:

$$\vec{R}(t) \;=\; \vec{R}_0 + t\vec{v}_0 \quad\Longleftrightarrow\quad \begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \\ z(t) &= z_0 + ct \end{aligned} \quad\Longleftrightarrow\quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad \text{if } abc \neq 0$$

▷ The plane $\mathcal{P}$ through $R_0 = (x_0,\ y_0,\ z_0)$ and with normal $\vec{n} = \langle a,\ b,\ c\rangle$:

$$\left(\vec{R} - \vec{R}_0\right) \bullet \vec{n} = 0 \quad\Longleftrightarrow\quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Longleftrightarrow\quad ax + by + cz = d \text{ where } d = ax_0 + by_0 + cz_0\,.$$

▷ For a plane $\mathcal{P}$ given by $ax + by + cz = d$:

$$d\left(\mathcal{P}, (x_1, y_1, z_1)\right) \;=\; \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

▷ For radius ρ, center R_0, and a right-handed system of orthonormal vector $\vec{e}_1, \vec{e}_2, \vec{e}_3$:

$$\vec{R}(t) \;=\; \vec{R}_0 + \rho\cos t\,\vec{e}_1 + \rho\sin t\,\vec{e}_2 + at\,\vec{e}_3$$

- ∘ if $a = 0$: is a circle in the plane determined by $\vec{e}_1$ and $\vec{e}_2$
- ∘ if $a > 0$: is a right–handed helix with pitch $2\pi\,|a|$ and axis parallel to $\vec{e}_3$
- ∘ if $a < 0$: is a left–handed helix with pitch $2\pi\,|a|$ and axis parallel to $\vec{e}_3$

1.7. Force: For a constant force $\vec{F}$ acting thru a displacement $\vec{D}$

▷ The work W done by $\vec{F}$ is $\qquad W = \vec{D} \bullet \vec{F}$

▷ The torque τ due to $\vec{F}$ at the point D is $\qquad \tau = \vec{D} \times \vec{F}$.

1.8. Cylindrical Coordinates (r, θ, z) of (x, y, z) :

$$r \geqslant 0 \qquad 0 \leqslant \theta < 2\pi \qquad -\infty < z < \infty$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \text{ and } y \geqslant 0 \\ \pi + \tan^{-1}(y/x) & \text{if } x < 0 \\ 2\pi + \tan^{-1}(y/x) & \text{if } x > 0 \text{ and } y < 0 \end{cases} \qquad z = z$$

1.9. Spherical Coordinates (ρ, θ, ϕ) of (x, y, z) :

$$\rho \geqslant 0 \qquad 0 \leqslant \theta < 2\pi \qquad 0 \leqslant \phi \leqslant \pi$$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \text{as above} \qquad \phi = \cos^{-1}\frac{z}{\rho}$$