MATH 544
Prof. Meade

University of South Carolina
Spring 2000

Final Exam
May 2, 2000

Name: ______________________
SS #: ______________________

Instructions:

1. There are a total of 8 problems on 9 pages. Check that your copy of the exam has all of the problems.
2. You must **show all of your work** to receive credit for a correct answer.
3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	25	
2	15	
3	20	
4	10	
5	20	
6	30	
7	15	
8	15	
Total	150	

Have a Great Summer!

1. (25 points) Assume the matrices A and B are row equivalent.

$$A = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 2 & -4 & 9 & -8 & 12 & 6 \\ 3 & -6 & 12 & -9 & 17 & 10 \\ -5 & 10 & -9 & -7 & -15 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\operatorname{Nul} A$.

(b) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$ in R^4? (Why or why not?)

2. (15 points) Let A be an $n \times n$ matrix. Which of the following statements is equivalent to the statement that A is invertible.

_______ the equation $A\mathbf{x} = \mathbf{b}$ has no solution for some $\mathbf{b}$

_______ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for $\mathbf{b} = \mathbf{0}$

_______ $\det A > 0$

_______ $\det A = 0$

_______ $\det A \neq 0$

_______ A is row equivalent to the identity matrix

_______ A is row equivalent to a matrix with a row of zeroes

_______ A is row equivalent to the zero matrix

_______ A has n distinct eigenvalues

_______ A is diagonalizable

_______ $\lambda = 0$ is an eigenvalue of A

_______ $\lambda = 0$ is not an eigenvalue of A

_______ $\operatorname{rank} A < n$

_______ $\operatorname{rank} A = n$

_______ $\operatorname{rank} A > n$

3. (20 points) Let A be a 12×5 matrix.

(a) What is the largest possible value for $\operatorname{rank} A$?

(b) What is the smallest possible value for $\operatorname{rank} A$?

(c) What is the largest possible value for $\dim \operatorname{Nul} A$?

(d) What is the smallest possible value for $\dim \operatorname{Nul} A$?

(e) What is the size of $A^T A$?

(f) Use the Rank Theorem to express $\operatorname{rank} A$ in terms of $\dim \operatorname{Nul} A$.

(g) Use the Rank Theorem and the fact that $\operatorname{Nul}(A^T A) = \operatorname{Nul} A$ to express $\operatorname{rank}(A^T A)$ in terms of $\dim \operatorname{Nul} A$.

(h) What is the relationship between $\operatorname{rank} A$ and $\operatorname{rank}(A^T A)$?

(i) Suppose the columns of A are linearly independent. Explain why $A^T A$ is invertible.

4. (10 points) Consider the linear transformation $T : \mathrm{R}^3 \to \mathrm{R}^2$ that maps the standard basis vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ onto $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 9 \\ 3 \end{bmatrix}$.

   (a) What is $T(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix})$?

   (b) Find the matrix A of the linear transformation.

   (c) Find all solutions to $T(\mathbf{x}) = \mathbf{0}$.

5. (20 points) Let $A = \begin{bmatrix} 0 & 4 & 2 \\ 4 & 0 & -2 \\ 2 & -2 & 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors for A.

(a) What are the eigenvalues of A?

(b) Find matrices P and D that diagonalize A and write the equation that relates A to P and D.

6. (30 points) Let $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 6 \\ -3 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) Determine whether $\mathbf{x}$ is in W.

(b) Find a basis for W.

(c) What is the dimension of W?

(d) What is the dimension of $W^\perp$, the orthogonal complement of W.

(e) Determine whether $\mathbf{y}$ is in $W^\perp$.

7. (15 points) Find an orthogonal basis for the matrix

$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

8. (15 points)

   (a) Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line for the data $(-1, 0)$, $(0, 1)$, $(-1, 1)$, $(1, 2)$, and $(2, 4)$.

   (b) Suppose an experiment produces (x, y) data $(2, 5)$, $(3, 6)$, $(4, 8)$, $(5, 11)$ and a scientist wants to model this data with an equation of the form $y = \beta_1 x + \beta_2 x^2 + \beta_3 e^x$. Write the design matrix and the observation vector for this problem. What is the unknown parameter vector?