Homework Solutions

§7.5

#1. Claim: Let A & B be $n \times n$ matrices. If λ is a nonzero eigenvalue of AB then λ is an eigenvalue of BA.

Proof: Because λ is an eigenvalue of AB there is a nonzero vector x such that $ABx = \lambda x$. Now, multiply this by B to find $BABx = \lambda Bx$. Let $v = Bx$ and this can be rewritten as $BAv = \lambda v$. This means λ is an eigenvalue of BA with eigenvector $v = Bx$ provided $v \neq 0$.

If $v = Bx = 0$ then the original equation $ABx = \lambda x$ reduces to $Av = \lambda x$ and so $0 = \lambda x$. This cannot be satisfied if $\lambda \neq 0$ (because $x \neq 0$) and so $v \neq 0$ (by contradiction) and v is an eigenvector of BA.

$\boxed{}$

#3. Claim: Suppose A is an $n\times n$ matrix. If A has an eigenvalue with geometric multiplicity n then A is a multiple of the identity matrix.

Proof: Let λ be the eigenvalue of A with geometric multiplicity n. This means the nullspace of $A-\lambda I$ has dimension n. Because

$$\dim(\mathrm{col}(A-\lambda I)) + \dim(\mathrm{nul}(A-\lambda I)) = n$$

we see that $\dim(\mathrm{col}(A-\lambda I)) = 0$. Now, the only vectorspace with dimension zero is the zero vector. So every column of $A-\lambda I$ is the zero vector. That is,

$$A-\lambda I = 0 \text{ so } A = \lambda I. \ \boxed{}$$

#4. Claim: Let A & B be $n\times n$ matrices that are both diagonalized by the same matrix P.

Then $AB = BA$.

Proof: If A is diagonalized by P then $P^{-1}AP = K_1$

If B —— " —————— $P^{-1}BP = K_2$

where K_1, K_2 are diagonal matrices.

Then $A = PK_1P^{-1}$ and $B = PK_2P^{-1}$.

$$AB = (PK_1P^{-1})(PK_2P^{-1}) = PK_1K_2P^{-1}$$

and

$$BA = (PK_2P^{-1})(PK_1P^{-1}) = PK_2K_1P^{-1}$$

Because diagonal matrices commute: $K_1K_2 = K_2K_1$

and so $AB = P(K_1K_2)P^{-1} = P(K_2K_1)P^{-1} = BA. \ \boxed{}$