Homework Solutions

§7.4

#1. (a) $A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = -2, \ \lambda_2 = 1$ so $K = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda = -2$: $A-(-2)I = \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ so $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 1$: $A - I = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix}$ so sol'ns to $(A-I)u=0$ are $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This gives us that A is diagonalized by $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ has e-values 3 & 0 so $K = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$.

$\lambda = 3$: $A - 3I = \begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$ so $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda = 0$: $A - 0I = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ has null space basis $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This matrix is also diagonalized by $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix}$ has eigenvalues $-2, 1$ so $K = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.

$\lambda = -2$: $A-(-2)I = \begin{bmatrix} 0 & 0 \\ -6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ so $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\lambda = 1$: $A - I = \begin{bmatrix} -3 & 0 \\ -6 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$ so $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

This matrix is diagonalized by $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

#2. $M = \begin{bmatrix} 16 & -35 \\ 6 & -13 \end{bmatrix}$ has e-vectors $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

(a) M is diagonalized by $P = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$.

To find the eigenvalues of P we can compute (as in §7.1)

$Mu = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2u$ so this e-vector has $\lambda = 2$.

$Mv = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 7 \\ 3 \end{bmatrix} = 1u$ $\lambda = 1$.

Thus, $K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and we can write $M = PKP^{-1}$

(b) $M^3 = (PKP^{-1})^3 = PK^3P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 24 & -56 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 106 & -245 \\ 42 & -97 \end{bmatrix}$

(c) $M^{-1} = (PKP^{-1})^{-1} = PK^{-1}P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3/2 & -7/2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -13/2 & 35/2 \\ -3 & 8 \end{bmatrix}$

#5. Claim: If A is similar to B and A is singular, then B must also be singular.

Proof: If A is singular then at least one of A's eigenvalues must be zero.

Because A & B are similar, they have the same eigenvalues.

Thus, at least one of B's eigenvalues will also be zero which means B is singular. $\boxed{\text{S}}$

#6. (a) A diagonalizable means A has a basis of eigenvectors.

A invertible means A has non-zero eigenvalues.

If A has a zero eigenvalue and a basis of eigenvectors then A is diagonalizable and not invertible.

False

(b) Similar to (a) except now if A has all eigenvalues non-zero and is deficient, then A is invertible & not diagonalizable.

False.

(c) If λ is an eigenvalue of A with algebraic multiplicity n it is diagonalizable only when λ also has geometric multiplicity n. Since this is not guaranteed, this statement is false.

(d) If A is diagonalized by P, then

$$A = PKP^{-1} \quad \text{and} \quad A^2 = PK^2P^{-1}$$

so that A^2 is also diagonalized by P. False

(e) If A has n distinct eigenvalues, then each of them eigenspaces has dimension 1 and there is a basis of eigenvectors. This means A can be diagonalized. True.

#8. $H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

$H - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only 1 free variable so the geometric multiplicity is 1.

Geometric reason: H cannot be diagonalized because H turns all vectors not on the x-axis to the right (Hv is not parallel to v when v is not on the x-axis).

* Use Maple: H := Matrix([[1,1],[0,1]]);

Headtail(H);