# Homework Solutions

## §7.2

#1 (a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ $\quad p(\lambda) = \det \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) + 2$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

The e-values of A are $\lambda = 2$ & $\lambda = 3$, both with algebraic multiplicity 1.

(b) $B = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$ $\quad p(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{bmatrix} = (2-\lambda)(6-\lambda) + 4$

$$= \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$$

The e-value of A is $\lambda = 4$, with algebraic multiplicity 2.

(c) $C = \begin{bmatrix} 3 & -2 & +2 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$$p(\lambda) = \det \begin{bmatrix} 3-\lambda & -2 & +2 \\ 0 & 1-\lambda & 0 \\ -1 & 1 & -\lambda \end{bmatrix} = +(1-\lambda) \det \begin{bmatrix} 3-\lambda & +2 \\ -1 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)\Big((3-\lambda)(-\lambda) + 2\Big) = (1-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (1-\lambda)(\lambda-2)(\lambda+1) = -(\lambda-1)^2(\lambda-2)$$

The eigenvalues of C are $\lambda = 1$ (w/ alg. mult. 2) and $\lambda = 2$ (w/ alg. mult. 1).

(d) $D = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad p(\lambda) = \det \begin{bmatrix} 4-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda)(3-\lambda)$

The e-values of D are $\lambda = 2, \lambda = 3, \lambda = 4$ (each w/ alg. mult. 1).

#2 (a) $\lambda_1 = 2$: $A-2I = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad -x_1 - x_2 = 0$, $x_1 = -x_2$

solns to $(A-2I)x=0$ are $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The geom. mult of $\lambda_1 = 2$ is 1 & a basis for the e-space is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$: $A-3I = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad -2x_1 - x_2 = 0$, $x_1 = -\frac{1}{2}x_2$

solns to $(A-3I)x=0$ are $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

The geom. mult. of $\lambda_2 = 3$ is 1 & a basis for the e-space is $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

(b) $\lambda_1 = 4$: $B-4I = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad -2x_1 + x_2 = 0$, $x_1 = \frac{1}{2}x_2$

solns to $(B-4I)x=0$ are $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

The geom. mult of $\lambda = 4$ is 1 & a basis for this e-space is $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

(c) $\lambda = 1$: $C-I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad 2x_1 = 2x_2 - 2x_3$, $x_1 = x_2 - x_3$

solns to $(C-I)x=0$ are $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The geom. mult. of $\lambda = 1$ is 2 & a basis for the e-space is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = 2$: $C-2I = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 0 \\ -1 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

solns to $(C-2I)x=0$ are $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

The geom. mult. of $\lambda = 2$ is 1 & a basis for the e-space is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\lambda_1 = 2$: $D - 2I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $2x_1 + x_2 = 0$ $x_1 = -\frac{1}{2}x_2$, $x_3 = 0$

sol'ns to $(D-2I)x = 0$ are $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$

the geom. mult. of $\lambda = 2$ is 1 & a basis for the e-space is $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$.

$\lambda_2 = 3$: $D - 3I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_2 - x_3 = -2x_3$, $x_2 = x_3$

sol'ns to $(D-3I)x = 0$ are $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

the geom. mult. of $\lambda = 3$ is 1 & a basis for the e-space is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda_3 = 4$: $D - 4I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = x_1$, $x_2 = -x_3 = 0$, $x_3 = 0$

sol'ns to $(D-4I)x = 0$ are $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

the geom. mult. of $\lambda = 4$ is 1 & a basis for the e-space is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

#3. $p(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 = \lambda^2(\lambda^2 - 5\lambda + 6) = \lambda^2(\lambda - 2)(\lambda - 3)$.
The matrix $A$ is $4 \times 4$ and has e-values $\lambda_1 = 0$ (alg. mult. 2), $\lambda_2 = 2$ (alg. mult 1) and $\lambda_3 = 3$ (alg. mult. 1).

#5. (a) Because the matrix is lower-triangular the eigenvalues are the diagonal elements. Choose 2 of $a, c, f$ to be 3 and the third variable to be $-5$ and $b, d, e$ can be anything.

(b). Not possible since $a, c, f$ must be 3.

#7. The picture indicates that the 2x2 matrix has 3 eigenvalues. A 2x2 matrix can have only 2 e-values.

#8. When an eigenvalue has algebraic multiplicity 1 the geometric multiplicity must also be 1. So each of the three eigenspaces is a line through the origin.

#9. The 3x3 matrix $A$ has eigenvalues $\lambda_1 = 5$ with eigenvector $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\lambda_2 = 2$ (alg. multiplicity 2) and eigenvector $\begin{bmatrix}0\\1\\0\end{bmatrix}$. (geom. mult. 1).

#12. $A = \begin{bmatrix}1 & k\\ k & 1\end{bmatrix}$ $\quad p(\lambda) = \det\left(\begin{bmatrix}1-\lambda & k\\ k & 1-\lambda\end{bmatrix}\right) = (1-\lambda)^2 - k^2 = 0$

$$(1-\lambda)^2 = k^2$$
$$1-\lambda = \pm k.$$
$$\lambda = 1 \pm k.$$

(a) $A$ has 2 distinct eigenvalues whenever $k \neq 0$.

The two eigenvalues are $\lambda_1 = 1+k$ & $\lambda_2 = 1-k$.

(b) $k=1$: $A = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ $\quad p(\lambda) = \det\left(\begin{bmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{bmatrix}\right) = (1-\lambda)^2 - 1$

$$= 1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda$$
$$= \lambda(\lambda - 2)$$

The eigenvalues are $\lambda_1 = 0$ & $\lambda = 2$.

These are the predicted eigenvalues $\lambda = 1 \pm 1 = 0$ & $2$.