Homework Solution

§ 5.7

#1. $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 4 \\ 3 & -1 \end{bmatrix}$

Observe that $2A + B - C = 0$.

#4. $W = \{$upper-triangular 2×2 matrices$\}$.

Claim: W is a subspace of M_{22}.

Proof: 1. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

2\. let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$.

then $A+B = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} \in W$.

3\. let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$; then $dA = \begin{bmatrix} da & db \\ 0 & dc \end{bmatrix} \in W$.

so W is a subspace of M_{22}. $\square$

#7. (a) $W = \{$upper- or lower-triangular matrices in $M_{22}\}$.

not a subspace of M_{22} because W is not closed under addition.

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ are both in W

but $A+B = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \notin W$.

(b) $W = \{$ 2×2 matrices with non-negative entries $\}$.

if $A \in W$ then $-2A \notin W$ (because all entries of $-2A$ are non-positive!)

so W is not closed under scalar multiplication.

(d) $W = \{ 2\times 2 \text{ matrices with } A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \}$.

Let A & B be two matrices in W, i.e. $A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$B\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$.

then $(A+B)\begin{bmatrix}2\\3\end{bmatrix} = A\begin{bmatrix}2\\3\end{bmatrix} + B\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\end{bmatrix}$

so that $A+B \notin W$.

(Oh - it's even easier: $\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\end{bmatrix}$ so $0 \notin W$!)

#11. $W = \{ \text{matrices } A \text{ in } M_{32} \text{ with } A\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \}$.

(a) W contains 0 & is closed under addition & scalar multiplication.

(b) The matrices in W have the form $A = \begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix}$

where
$$2a+3d=0 \Rightarrow d = -\tfrac{2}{3}a$$
$$2b+3e=0 \Rightarrow e = -\tfrac{2}{3}b$$
$$2c+3f=0 \Rightarrow f = -\tfrac{2}{3}c.$$

This means the general form of A is

$$A = \begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix} = \begin{bmatrix}a&-\frac{2}{3}a\\b&-\frac{2}{3}b\\c&-\frac{2}{3}c\end{bmatrix} = a\begin{bmatrix}1&-\frac{2}{3}\\0&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&0\\1&-\frac{2}{3}\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\0&0\\1&-\frac{2}{3}\end{bmatrix}$$

From this decomposition we ~~see~~ see that W is spanned by

$$\begin{bmatrix}1&-\frac{2}{3}\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&-\frac{2}{3}\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&-\frac{2}{3}\end{bmatrix},$$

and so $\dim(W) = 3$.

#16. (a) To show $\{2+x^2,\ 1+2x,\ (2+x)^2\}$ is linearly dependent we have to find a nontrivial solution to

$$c_1(2+x^2)+c_2(1+2x)+c_3(2+x)^2=0$$
$$c_1(2+x^2)+c_2(1+2x)+c_3(4+4x+x^2)=0$$
$$(2c_1+c_2+4c_3)+(2c_2+4c_3)x+(c_1+c_3)x^2=0$$

This requires
$$2c_1+c_2+4c_3=0 \implies -2c_3-2c_3+4c_3=0 \text{ or } 0=0.$$
$$2c_2+4c_3=0 \implies c_2=-2c_3$$
$$c_1+c_3=0. \implies c_1=-c_3$$

So, Let $c_3=1$ & set $c_1=-1$, $c_2=-2$ and we have a nontrivial solution.

Check: $-(2+x^2)-2(1+2x)+(2+x)^2 = -2-x^2-2-4x+4+4x+x^2=0.$

(c) To show $\{\sin^2 x, \cos^2 x, 1\}$ is linearly dependent, note that $\sin^2 x+\cos^2 x-1=0$.

(d) To show $\{\sin(x+\pi/4), \sin x, \cos x\}$ is linearly dependent note that

$$\sin(x+\pi/4)=\sin x\cos\pi/4+\cos x\sin\pi/4=\frac{\sqrt2}{2}\sin x+\frac{\sqrt2}{2}\cos x.$$

i.e. $\sin(x+\pi/4)$ can be expressed as a linear combination of $\sin x$ & $\cos x$.

#17. We already know $\{\sin^2 x, \cos^2 x, 1\}$ is lin. dependent.

Because $\sin^2 x + \cos^2 x - 1 = 0$ we see that any one of the 3 elements of the set can be removed without changing the span of the remaining 2 functions.

This gives us 3 possible bases:

$\{\sin^2 x, \cos^2 x\}$

$\{\sin^2 x, 1\}$

$\{\cos^2 x, 1\}$

because each pair of functions is not parallel, so they are independent and therefore a basis

The Span $\{\sin^2 x, \cos^2 x, 1\}$ is a 2-dimensional subspace of $F(\mathbb{R}, \mathbb{R})$.

#18. (a) a basis for P_2 must have 3 elements ($\dim(P_2) = 3$)

so this set of 2 functions cannot span P_2.

(b) These 3 functions are a basis provided they are independent.

To check independence we find all solutions to

$$c_1(1+x) + c_2(1+x^2) + c_3(x+x^2) = 0.$$

Group by powers of x: $(c_1+c_2)1 + (c_1+c_3)x + (c_2+c_3)x^2 = 0$.

So we must have:
$$\begin{aligned} c_1+c_2 &= 0 \\ c_1+c_3 &= 0 \\ c_2+c_3 &= 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This brings the problem back to familiar territory.

Using Maple, we find that the only solution to is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

This means $c_1 = c_2 = c_3 = 0$ & there are no linear dependence relations.

The functions $1+x$, $1+x^2$, & $x+x^2$ are a basis for P_2.

(c) We proceed in the same manner as in (b).

$$c_1(1+x)+c_2(1+2x+x^2)+c_3(x+x^2)=0$$

$$(c_1+c_2)1+(c_1+2c_2+c_3)x+(c_2+c_3)x^2=0$$

So
$$\begin{aligned} c_1+c_2&=0\\ c_1+2c_2+c_3&=0\\ c_2+c_3&=0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

It is possible to see that these three columns are dependent

$$\begin{bmatrix} 1\\ 1\\ 0 \end{bmatrix}-\begin{bmatrix} 1\\ 2\\ 1 \end{bmatrix}+\begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}.$$

(If you don't see this, find all sol'ns to the above system and you will see this linear dependence relation.)

The bottom line is that these three functions are linearly dependent and so cannot span P_2 and are not a basis for P_2.

(d) No set of 4 functions in P_2 can be linearly independent.
These functions are not a basis for P_2.