# Homework Solutions

## §5.5

#1. (a) True (Theorem 3.6)

(b) False. Rowspace & column space have the same dimension but it doesn't have to be 4.

(c) True (Theorem 7)

(d) False. The columns of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ are independent but the rows are dependent.

#4. $A \in \mathbb{R}^{m\times n}$, rank $A = r$.

(a) $\dim(\text{col}(A)) = \text{rank } A = r$

(b) $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A) = r$.

(c) $\dim(\text{nul}(A)) = n - r$.

(d) $\dim(\text{nul}(A^T)) = m - r$ (because $A^T \in \mathbb{R}^{n\times m}$)

(e) # of free var's in $Ax=0$ $= \dim(\text{nul}(A)) = n-r$

(f) # of free vars in $A^Tx=0$ $= \dim(\text{nul}(A^T)) = m-r$.

#5. (a) $A \in \mathbb{R}^{5\times 6}$, $\text{rank}(A) = 4$ : $\dim(\text{nul}(A)) = 2$

$\dim(\text{row}(A)) = \dim(\text{col}(A)) = 4$

$\dim(\text{col}(A)) = \text{rank}(A) = 4$.

(b) $A \in \mathbb{R}^{5\times 4}$, $\dim(\text{nul}(A)) = 1$ : $\dim(\text{row}(A)) = \dim(\text{col}(A)) = n-1 = 3$.

(c) $A \in \mathbb{R}^{6\times 5}$, $\dim(\text{row}(A)) = 3$ : $\dim(\text{col}(A)) = \dim(\text{row}(A)) = 3$

$\dim(\text{nul}(A)) = 5-3 = 2$

(d) $A \in \mathbb{R}^{5\times 7}$, $\dim(\text{nul}(A^T)) = 2$ : $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \dim(\text{col}(A^T))$

$= 5-2 = 3$

$\dim(\text{nul}(A)) = 7-3 = 4$.

#6. $A \in \mathbb{R}^{m \times n}$.

$\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A))$.

Now, the row space dimension cannot be larger than the # of columns: $\text{rank}(A) \leq n$

And, the column space dimension cannot be larger than the # of rows: $\text{rank}(A) \leq m$.

Combined, $\text{rank}(A) \leq m$ & $\text{rank}(A) \leq n$ we see that the rank of a matrix cannot exceed the smaller of $m$ & $n$. $\boxed{}$

#8. $A \in \mathbb{R}^{m \times n}$ with $m < n$.

The echelon form of A has to have at least $n-m$ free variables. Since $n-m > 0$ the null space of A is not $\{0\}$.

#13.