Homework Solution

§5.3

#1. (a) True, 4 vectors in a 3-dim. vector space must be dependent.

(b) False. If $v_1 = v_2 = v_3 = v_4$ then $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathrm{Span}\{v_1\}$ is a 1-dim. subspace of $\mathbb{R}^3$.

(c) False. Some subset of 3 vectors will be a basis but you don't know that v_4 is the vector you can discard.

e.g. $v_1 = e_1,\ v_2 = e_2,\ v_3 = e_2,\ v_4 = e_3$.

(d) True. You can find one of the vectors to omit and have a minimal spanning set.

#3. Let $\{v_1, \dots, v_k\}$ be a basis for W_2. Because $W_1 \subset W_2$, a subset of these vectors will be a basis for W_1. Thus,

$$\dim W_1 \leq \dim W_2.$$

#5. This is similar to §5.2 #15. We know $\{v_1, \dots, v_n\}$ is a linearly independent set in $\mathbb{R}^n$ and want to conclude that $\{Av_1, \dots, Av_n\}$ is also linearly independent (assuming A is invertible). Consider

$$d_1(Av_1) + d_2(Av_2) + \dots + d_n(Av_n) = A(d_1 v_1 + d_2 v_2 + \dots + d_n v_n) = 0.$$

Because A is invertible the only soln to $Ax = 0$ is $x = 0$. So, implies that $d_1 v_1 + d_2 v_2 + \dots + d_n v_n = 0$ and this is true only when $d_1 = d_2 = \dots = d_n = 0$. So, $\{Av_1, Av_2, \dots, Av_n\}$ must be linearly independent. Because there are n of these vectors they must also span $\mathbb{R}^n$, hence they are a basis for $\mathbb{R}^n$.

#7. (a) $W_1 = \text{Span}\{u_1, \dots, u_j\}$ and $W_2 = \text{Span}\{v_1, \dots, v_k\}$

with $W_1 \cap W_2 = \{0\}$.

To show that $\{u_1, \dots, u_j, v_1, \dots v_k\}$ is lin. independent, look at

$$c_1 u_1 + \dots + c_j u_j + d_1 v_1 + \dots + d_k v_k = 0$$

$$\Leftrightarrow c_1 u_1 + \dots + c_j u_j = -(d_1 v_1 + \dots + d_k v_k)$$

Now, the only way a linear combination of the u_i can be equal to a linear combination of the v_i is when both linear combinations are zero:

$$c_1 u_1 + \dots + c_j u_j = 0 \longrightarrow c_1 = c_2 = \dots = c_j = 0$$
$$d_1 v_1 + \dots + d_k v_k = 0 \implies d_1 = d_2 = \dots = d_k = 0.$$

So the set of $j+k$ vectors is linearly independent.

(b) $W_1 = \text{Span}\{u_1\}$, $W_2 = \text{Span}\{v_1, v_2\}$, with $W_1 \cap W_2 = \{0\}$.

Then $\{u_1, v_1, v_2\}$ is a linearly independent set of vectors $\mathbb{R}^3$. Thus $\text{Span}\{u_1, v_1, v_2\}$ has dimension 3, i.e. $\text{Span}\{u_1, v_1, v_2\} = \mathbb{R}^3$.

$$\dim W_1 + \dim W_2 = \dim(\mathbb{R}^3)$$