Homework Solutions

§5.2

#1. (a) these 2 vectors are parallel, not independent, not a basis.

(b) these 2 vectors in $\mathbb{R}^2$ are not parallel, they are independent & they are a basis for $\mathbb{R}^2$

(c) these 3 vectors in $\mathbb{R}^3$ must be dependent & cannot be a basis.

#4. (a) $W = \left\{ \begin{bmatrix} 2t \\ -5t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \right\}$; $\dim W = 1$.

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y + 4z = 0 \right\} = \left\{ \begin{bmatrix} x \\ 2x+4z \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \right\}$ $\dim W = 2$.

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y + 4z = 0 \;\&\; x + 3z = 0 \right\} = \left\{ \begin{bmatrix} -3z \\ -2z \\ z \end{bmatrix} \right\}$ $z \in \mathbb{R}$

$= \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$ $\dim W = 1$

#6. (a) $v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

Let $v_3 = v_1 \times v_2$ to ensure $v_3 \perp v_1$ & $v_3 \perp v_2$.

$$v_3 = v_1 \times v_2 = \det \begin{bmatrix} i & j & k \\ 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \langle -1, -2, 6 \rangle = \begin{bmatrix} -1 \\ -2 \\ 6 \end{bmatrix}.$$

(b) Geometrically, all you need is to ensure is that v_3 is not in the plane spanned by v_1 & v_2.

#10. $W = \left\{ \begin{bmatrix} -x_2+2x_5 \\ x_2 \\ x_4-3x_5 \\ x_4 \\ x_5 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ 2

A basis for W is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$; $\dim W = 3$.

#11. (a) True. By Thm 5(c), 4 lin. indep vectors in $\mathbb{R}^4$ must also span $\mathbb{R}^4$.

(b) True. Every set of lin. indep. vectors in $\mathbb{R}^4$ is a basis for its span.

(c) False. There is no guarantee that $\{v_1, v_2, v_3\}$ is a linearly independent set of vectors.

(d) True. If 3 vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$ then they must be linearly independent.

(e) True. 5 vectors in $\mathbb{R}^4$ must be lin. dependent.

(f) False. Consider the case when $v_4 = 0$ & v_1, v_2, v_3 are linearly independent. Then $\text{Span}\{v_1, v_2, v_3, v_4\} = \text{Span}\{v_1, v_2, v_3\}$ which has dimension 3.

(g) False: While there is a basis formed from 2 of v_1, v_2, v_3, there is no guarantee that v_1 & v_2 are linearly independent.

#13. (a) basis: $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$, dimension $= 1$ $\quad (v_2 = -3v_1)$

(b) basis: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$, dimension $= 2$ $\quad (v_3 = v_2 - v_1)$.

(c) basis: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$, dimension $= 3$.

#15. Because $\{u, v, w\}$ is a basis for W we know $c_1u + c_2v + c_3w = 0$ only when $c_1 = c_2 = c_3 = 0$. Also, $\dim W = 3$.

(a) Because we have 3 vectors, if they are linearly independent they also span W. To check linear independence of $\{2u, -3v, w\}$:

$$d_1(2u) + d_2(-3v) + d_3w = (2d_1)u + (-3d_2)v + d_3w = 0$$

only when $2d_1 = 0$, $-3d_2 = 0$ & $d_3 = 0$, i.e. when $d_1 = d_2 = d_3 = 0$.

Since $\{2u, -3v, w\}$ is lin. indep & spans W, it is a basis for W.

(b) Following the approach in (a):

$$d_1(u+v) + d_2(u+w) + d_3(v+w) = (d_1+d_2)u + (d_1+d_3)v + (d_2+d_3)w = 0$$

only when
$d_1 + d_2 = 0 \Rightarrow d_2 = -d_1$
$d_1 + d_3 = 0 \Rightarrow d_3 = -d_1$
$d_2 + d_3 = 0$
$\Longrightarrow -d_1 + d_3 = -2d_1 = 0 \Rightarrow d_1 = 0$.

Thus, $d_2 = d_3 = -d_1 = 0$ & so these 3 vectors are linearly independent and therefore a basis for W.

(c) Repeating the process from (a) & (b):

$$d_1(u+v) + d_2(u+w) + d_3(v-w) = (d_1+d_2)u + (d_1+d_3)v + (d_2-d_3)w = 0$$

only when
$d_1 + d_2 = 0 \rightarrow d_2 = -d_1$
$d_1 + d_3 = 0 \rightarrow d_3 = -d_1$
$d_2 - d_3 = 0$
$\Rightarrow -d_1 - (-d_1) = 0$

A nontrivial solution is $d_1 = -1$, $d_2 = d_3 = 1$. so these vectors are linearly dependent and <u>not</u> a basis for W.

(d) There are only 2 vectors; these cannot span the 3 dimensional vectorspace W.

#18. The blue dot is at $-9e_1 - 10e_2$, i.e. $(-9, -10)$ in Cartesian coordinates. Looking at the grid formed by $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, the same point is represented by

$-2u - 3v$, i.e. has coordinates $(-2, -3)$ in u-v coordinates.