Homework Solutions

§5.1

#1. (a) $W = \left\{ \begin{bmatrix} a \\ 2a \\ -3a \end{bmatrix} : a \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$.

(b) $W = \left\{ \begin{bmatrix} a+2b \\ 2a-b \\ -3a \end{bmatrix} : a,b \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$.

(c) $W = \left\{ \begin{bmatrix} a+2b+c \\ 2a-2c \\ -3a+b+4c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\}$.

#2. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does not satisfy $2x-2y+4z=5$ so $0 \notin$ subset.

(b) $\begin{bmatrix} a+b \\ a-b \\ b+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \left.\begin{matrix} a=-b=3 \\ a=b=-3 \\ b=-3 \end{matrix}\right\}$ not possible!

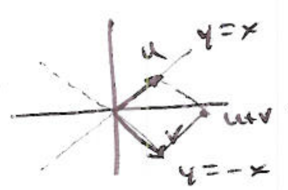
(c) if $u = \begin{bmatrix} a^2 \\ 2a \end{bmatrix}$ and $v = \begin{bmatrix} b^2 \\ 2b \end{bmatrix}$, then $u+v = \begin{bmatrix} a^2+b^2 \\ 2a+2b \end{bmatrix} = \begin{bmatrix} a^2+b^2 \\ 2(a+b) \end{bmatrix}$

$\neq \begin{bmatrix} (a+b)^2 \\ 2(a+b) \end{bmatrix}$ (provided $ab \neq 0$)

so this subset is not closed under addition

(d) if $u = \begin{bmatrix} m/n \\ m/n \end{bmatrix}$ with $m+n$ both integers

then $cu = \begin{bmatrix} cm/n \\ cm/n \end{bmatrix}$ is in the set except when c is irrational.

(e) 

← not on either line, so not closed under addition

(f) $A0 = 0 \neq b$ so 0 is not in this set.

#7 (a) Only the trivial subspace $\{0\}$ has exactly one vector.

(b) No subspace can have exactly 2 vectors. If there are two vectors in the subspace then the subspace must be either a line through the origin or a plane through the origin or the full space.

(c) Yes. The span of one nonzero vector gives a line.

(d) Yes. The span of any collections of appropriately sized vectors is a subspace. But these 4 vectors in $\mathbb{R}^3$ cannot be independent

#8. There are 4 basic subspaces of $\mathbb{R}^3$:

(a) the trivial subspace: $\{0\}$

(b) lines through the origin: $\text{Span}\{v\}$

(c) planes through the origin: $\text{Span}\{v, w\}$

(d) all of $\mathbb{R}^3$.

#9. $W = \{x \in \mathbb{R}^n : Ax = x\} = \{x \in \mathbb{R}^n : Ax - x = 0\} = \{x \in \mathbb{R}^n : (A - I)x = 0\}$

in this last form we see that W is the solutions set of the homogeneous equation $(A-I)x = 0$ and so is a subspace of $\mathbb{R}^n$.

#15. $T(x) = Ax$ with $A = \begin{bmatrix} 0 & e_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\left\{\begin{array}{l} T(e_1) = 0 \\ T(e_2) = e_2 \\ T(e_3) = 0 \end{array}\right\}$

The kernel of T is the Span of e_1 & e_3.

(because $T(e_1) = T(e_3) = 0$).

The range of T must be ~~Span~~ $\text{Span}\{e_2\}$.

#20. In echelon form A will have one free variable and 2 pivot entries:

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$$

To find a, b, c note that $A\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives

$$2 - 3a + b = 0 \Rightarrow 3a - b = 2 \quad (\text{choose } b = 0, \text{ then } a = 2/3).$$
$$-3 + c = 0 \Rightarrow c = 3$$

so $A = \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

#22. $A \in \mathbb{R}^{3\times 5}$, $B \in \mathbb{R}^{5\times 4}$ $(AB \in \mathbb{R}^{3\times 4})$

	A	B	AB
kernel	$\mathbb{R}^5$	$\mathbb{R}^4$	$\mathbb{R}^4$
range	$\mathbb{R}^3$	$\mathbb{R}^5$	$\mathbb{R}^3$.