Homework Solutions

§4.2

#1. Scaling: inverse is a scaling with the reciprocal of each scalar factor.

Shear: inverse is another shear in the same variable in the opposite direction by the same amount.

Rotation: inverse is a rotation by same angle in the opposite direction.

Reflection: inverse is the reflection itself

Projection: not invertible because multiple vectors are mapped to the same vector. (the kernel is not a single vector, 0)

#2. In vector form, parallel lines have the same direction vector.

Let L_1 be $x = p_1 + tv_1$ & let L_2 be $x = p_2 + tv_1$ (the same direction)

Let $T(x) = Ax$. Then the image of L_1 is the points

$$T(x) = T(p_1 + tv_1) = A(p_1 + tv_1) = (Ap_1) + t(Av_1).$$

& for line 2:

$$T(x) = T(p_2 + tv_1) = A(p_2 + tv_1) = (Ap_2) + t(Av_1).$$

So, both images are lines with the same direction vector,

i.e. both images are parallel lines.

#3. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (a) pts on x-axis are $\begin{bmatrix} x \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

(b) horizontal line: $x = \begin{bmatrix} t \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(x) = A\left(\begin{bmatrix} 0 \\ a \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 0 \\ a \end{bmatrix} + t\,A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} ka \\ a \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

because the direction vector is horizontal we see that the image is the horizontal line $y = a$.

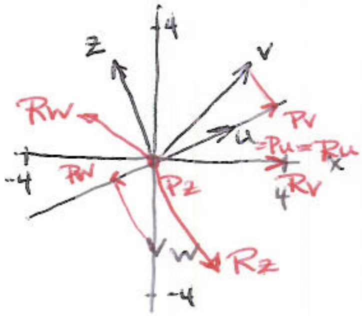
(c) vertical line: $x = \begin{bmatrix} a \\ t \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $T(x) = A\left(\begin{bmatrix} a \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} a \\ 0 \end{bmatrix} + t\,A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} a \\ 0 \end{bmatrix} + t\begin{bmatrix} k \\ 1 \end{bmatrix}$$

This is the line w/ x-intercept ($t=0$) a and direction vector $\begin{bmatrix} k \\ 1 \end{bmatrix}$,

so the slope is $1/k$.

#5.



#6. If A is the rotation matrix for a rotation by θ then A^2 should be the rotation matrix for a rotation by 2θ.

If A is a projection, then A^2 should be the same projection ($A^2 = A$).

If A is a reflection, then A^2 should be the identity ($A^2 = I$).

If A is a dilation by r, then A^2 should be a dilation by r^2.

#8. Let $x = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$. Rotation by an angle θ will be the vector $T(x) = \begin{bmatrix} r\cos(\varphi+\theta) \\ r\sin(\varphi+\theta) \end{bmatrix} = \begin{bmatrix} r\cos\varphi\cos\theta - r\sin\varphi\sin\theta \\ r\cos\varphi\sin\theta + r\sin\varphi\cos\theta \end{bmatrix}$

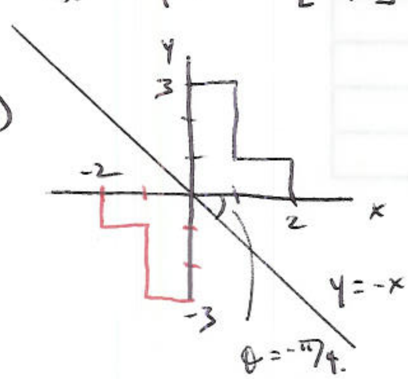
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix} = Rx$$

where $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

#10. (a) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ must be a reflection: $\left.\begin{matrix} \sin 2\theta = -1 \\ \cos 2\theta = 0 \end{matrix}\right\} \Rightarrow 2\theta = -\pi/2$ $\Rightarrow \theta = -\pi/4$

(b) Multiplication by A reflects the point $\begin{bmatrix} x \\ y \end{bmatrix}$ across the line $y = -x$ to the point $\begin{bmatrix} -y \\ -x \end{bmatrix}$.

(c)



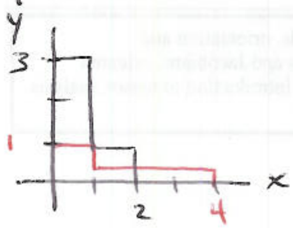
#10 (cont.)

$B = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$

(a) B is a scaling matrix with $r = 2$ & $s = 1/3$.

(b) When $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $Bu = \begin{bmatrix} 2x \\ y/3 \end{bmatrix}$ so the x-coord is doubled and the y-coord is divided by 3.

(c)

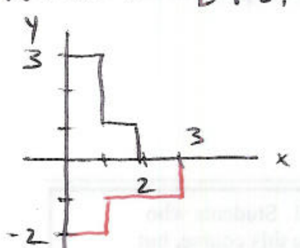


$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(a) C must be a rotation with $\cos\theta = 0$ & $\sin\theta = -1$, so $\theta = -\pi/2$.

(b) When $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $Cu = \begin{bmatrix} y \\ -x \end{bmatrix}$ is the rotation by an angle $\pi/2$ (90°) clockwise.

(c)

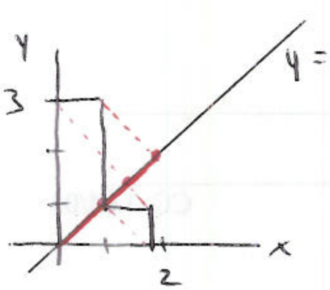


$D = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

(a) D must be a projection with $\sin^2\theta = \cos^2\theta = \sin\theta\cos\theta = 1/2$.
Either $\sin\theta = \cos\theta = 1/\sqrt{2}$ (so $\theta = \pi/4$) or $\sin\theta = \cos\theta = -1/\sqrt{2}$ $\left(\theta = -\frac{3\pi}{4}\right)$.

(b) When $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $Du = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$ is the projection onto the line $y = x$.

(c)



#11 (a) AB first applies B (reflection across the x-axis) followed by A (rotation by 90° counter-clockwise).

So ii is true

(b) A rotation followed by a reflection is not a rotation (in general).

Two reflections can be a rotation, so ii is true.

(c) Reflecting across the y-axis does not change the y component so the projection is unchanged, so ii could be true.

A rotation followed by a reflection will usually be different from a reflection followed by a rotation.

ii is true.