Homework Solution
§2.4

#2(a) $c_1\begin{bmatrix}1\\1\\4\end{bmatrix}+c_2\begin{bmatrix}-4\\2\\1\end{bmatrix}+c_3\begin{bmatrix}1\\4\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1&-4&1&0\\1&2&4&0\\4&1&4&0\end{bmatrix}\xrightarrow[r_3-4r_1\to r_3]{r_2-r_1\to r_2}\begin{bmatrix}1&-4&1&0\\0&6&3&0\\0&15&0&0\end{bmatrix}$$

$$\xrightarrow{r_3\leftrightarrow r_2}\begin{bmatrix}1&-4&1&0\\0&15&0&0\\0&6&3&0\end{bmatrix}$$

$$\xrightarrow{\frac{1}{15}r_2}\begin{bmatrix}1&-4&1&0\\0&1&0&0\\0&6&3&0\end{bmatrix}$$

$$\xrightarrow{r_3-6r_2\to r_3}\begin{bmatrix}1&-4&1&0\\0&1&0&0\\0&0&3&0\end{bmatrix}$$

unique solution $c_1=c_2=c_3=0$

$\therefore$ independent.

(b) $c_1\begin{bmatrix}1\\0\\1\end{bmatrix}+c_2\begin{bmatrix}-3\\4\\1\end{bmatrix}+c_3\begin{bmatrix}-1\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1&-3&-1&0\\0&4&2&0\\1&1&1&0\end{bmatrix}\xrightarrow{r_3-r_1\to r_3}\begin{bmatrix}1&-3&-1&0\\0&4&2&0\\0&4&2&0\end{bmatrix}\xrightarrow{r_3-r_2\to r_3}\begin{bmatrix}1&-3&-1&0\\0&4&2&0\\0&0&0&0\end{bmatrix}$$

There are an ∞ # of sol'ns $\Longrightarrow$ dependent

To find a lin. dep rel, backsolve:

$c_1-3c_2-c_3=0 \Longrightarrow c_1=3c_2+c_3=-\frac{3}{2}c_3+c_3=-\frac{1}{2}c_3$

$4c_2+2c_3=0 \Longrightarrow c_2=-\frac{1}{2}c_3$

so $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}c_3\\-\frac{1}{2}c_3\\c_3\end{bmatrix}$. Choose $c_3=-2$ to find $c_1=c_2=1$

$$1\begin{bmatrix}1\\0\\1\end{bmatrix}+1\begin{bmatrix}-3\\4\\1\end{bmatrix}+(-2)\begin{bmatrix}-1\\2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

(c) $c_1 \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{r_4 \leftrightarrow r_1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[r_4+3r_1 \to r_4]{r_2+2r_1 \to r_2} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[r_4-r_2 \to r_4]{r_3+r_2 \to r_3} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable(s) $\Rightarrow$ dependent.

Backsolve: $-c_1 + c_2 = 0 \Longrightarrow c_1 = c_2 = -\frac{1}{4}c_3$

$4c_2 + c_3 = 0 \Longrightarrow c_2 = -\frac{1}{4}c_3$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cancel{c_4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}c_3 \\ -\frac{1}{4}c_3 \\ c_3 \end{bmatrix}$ Choose $c_3 = -4 \Rightarrow c_1 = c_2 = +1$.

A linear dependence relation is:

$$1 \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

#3. (a) $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\a\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}1&2&0\\1&a&0\end{bmatrix} \xrightarrow{r_2-r_1\to r_2} \begin{bmatrix}1&2&0\\0&a-2&0\end{bmatrix}$$

unique sol'n only when $a-2=0$

$\underline{\underline{a=2}}$.

(b) $c_1\begin{bmatrix}1\\a\end{bmatrix} + c_2\begin{bmatrix}a\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}1&a&0\\a&4&0\end{bmatrix} \xrightarrow{r_2-ar_1\to r_2} \begin{bmatrix}1&a&0\\0&4-a^2&0\end{bmatrix}$$

unique sol'n only when $4-a^2=0$

$\underline{\underline{a=\pm 2}}$

(c) $c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\-1\\a\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1&1&1&0\\1&2&-1&0\\1&-1&a&0\end{bmatrix} \xrightarrow[r_3-r_1\to r_3]{r_2-r_1\to r_2} \begin{bmatrix}1&1&1&0\\0&1&-2&0\\0&-2&a-1&0\end{bmatrix} \xrightarrow{r_3+2r_2\to r_3} \begin{bmatrix}1&1&1&0\\0&1&-2&0\\0&0&a-5&0\end{bmatrix}$$

unique soln only when $a-5=0$ $\qquad \underline{\underline{a=5}}$.

#5. (a) if $\{v_1, v_2, v_3\}$ is a linearly dependent set in $\mathbb{R}^n$ then there is a linear dependence relation for these 3 vectors:

$c_1v_1+c_2v_2+c_3v_3=0$

This exact same equation is also a linear dependence relation for the set of 4 vectors $\{v_1, v_2, v_3, v_4\}$:

$c_1v_1+c_2v_2+c_3v_3+0v_4=0$.

So, $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent. $\underline{\underline{\text{True}}}$

(b) if $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ there there is no linear dependence relation for these vectors. if the only way to have $c_1v_1+c_2v_2+c_3v_3+c_4v_4=0$ is to have $c_1=c_2=c_3=c_4=0$, then $c_1v_1+c_2v_2+c_3v_3=0$ cannot have a nontrivial solution either. ~~False~~ $\underline{\underline{\text{True}}}$

(d) The equation $c_1v_1+c_2v_2+c_3v_3=0$ always has the trivial solution. This says nothing about the dependence or independence of any vectors. **False**

#8. $$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = c_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

Two different choices for the pair of free variables will give two different linear dependence relations. The easiest choices are:

1. $c_3=1,\ c_4=0$: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ $\quad \underline{2v_1+3v_2+v_3 \quad = 0}$

2. $c_3=0,\ c_4=1$: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underline{-v_1+5v_2 \quad +v_4 = 0}$