Homework Solutions

§1.1.

#1.b.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 6 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & 3 & 2 \\ & -8 & 0 \end{bmatrix}$$

Backsolve: $-8x_2 = 0 \implies \underline{x_2 = 0}$

$x_1 + 3x_2 = 2 \; : \; \underline{x_1 = 2}$

$\boxed{\text{Sol'n: } x_1 = 2, \; x_2 = 0}$

#2c.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 3 & 5 & 2 & 1 \end{bmatrix} \xrightarrow[r_3 - 3r_1 \to r_2]{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Backsolve: $-x_2 - x_3 = 1 \implies x_2 = -1 - x_3.$

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2(-1 - x_3) + x_3 = 0$

$x_1 - 2 - x_3 = 0$

$x_1 = 2 + x_3.$

Sol'n: $x_1 = 2 + x_3$, $x_2 = -1 - x_3$, $x_3 = x_3$

#3c.

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & -2 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3 - 2r_2 \to r_3} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

#6b. $\begin{bmatrix} 0 & 1 & 0 & 4 & 1 \\ 2 & 1 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_1} \begin{bmatrix} 2 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 & 1 \end{bmatrix}$ consistent (2 free vars.)

$$\begin{aligned} 2x_1 + x_2 \quad + x_4 &= 3 \\ x_2 \quad + x_4 &= 1 \end{aligned}$$

$x_4 = x_4$

$x_3 = x_3$

$x_2 = 1 - x_4$

$x_1 = \frac{1}{2}(3 - x_2 - x_4) = \frac{1}{2}(3 - (1 - x_4) - x_4) = \frac{1}{2}(2) = 1.$

so $x_1 = 1$

$x_2 = 1 - x_4$

$x_3 = x_3$

$x_4 = x_4.$

#7. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 8 \\ -1 & 1 & 5 \end{bmatrix} \xrightarrow[r_3 + r_1 \to r_3]{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 3 & 9 \end{bmatrix} \xrightarrow[r_3 + 3r_2 \to r_3]{} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 0 & -3 \end{bmatrix}$ not consistent (no soln)

#8. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & k & 2 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & k-6 & -10 \end{bmatrix}$ will be consistent provided $k - 6 \neq 0$.

#13 a) $\begin{bmatrix} 1 & 1 & 2 & 8 \\ 2 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -4 & -3 & -16 \end{bmatrix} \longrightarrow$

Backsolve: $x_1 + x_2 + 2x_3 = 8$ $\longrightarrow$ $x_1 = 8 - x_2 - 2x_3$

$-4x_2 - 3x_3 = -16 \Longrightarrow x_2 = 4 - \frac{3}{4}x_3$

$= 8 - (4 - \frac{3}{4}x_3) - 2x_3$

$= 4 - \frac{5}{4}x_3$

Soln: $x_1 = 4 - \frac{5}{4}x_3$

$x_2 = 4 - \frac{3}{4}x_3$

$x_3 = x_3$

b) $\begin{bmatrix} 3 & 1 & 2 & -1 & 8 \\ 2 & -2 & 1 & 2 & 0 \\ 1 & 3 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 2 & -2 & 1 & 2 & 0 \\ 3 & 1 & 2 & -1 & 8 \end{bmatrix}$

$\xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & -8 & -1 & 0 & 0 \\ 0 & -8 & -1 & -4 & 8 \end{bmatrix}$

$$\xrightarrow{r_3 - r_2 \to r_3} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & -8 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 8 \end{bmatrix}$$

Backsolve: $x_1 + 3x_2 + x_3 + x_4 = 0 \longrightarrow x_1 = -3x_2 - x_3 - x_4$

$= -3(-\frac{1}{8}x_3) - x_3 - (-2)$

$= 2 - \frac{5}{8}x_3$

$-8x_2 - x_3 = 0 \Rightarrow x_2 = -\frac{1}{8}x_3$

$-4x_4 = 8 \Rightarrow x_4 = -2$

Sol'n: $x_1 = 2 - \frac{5}{8}x_3$

$x_2 = -\frac{1}{8}x_3$

$x_3 = x_3$

$x_4 = 2$.

#15. (a) $\begin{bmatrix} 2 & 2 & k & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & k & 1 \\ 2 & 3 & 4 & 2 \end{bmatrix}$

$$\xrightarrow[r_3 - 2r_1 \to r_3]{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & k-2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{r_3 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & k-2 & 1 \end{bmatrix}.$$

(b) This system will be consistent when $k-2 \neq 0$. $(k \neq 2)$

(c) With $k=1$ the echelon form is $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & \cancel{k-2}\ -1 & 1 \end{bmatrix}$

Backsolve: $x_1 + x_2 + x_3 = 0 \longrightarrow x_1 = -x_2 - x_3 = -3$

$x_2 + 2x_3 = 2 \longrightarrow x_2 = 2 - 2x_3 = 2 + 2 = 4$

$-x_3 = 1 \Rightarrow x_3 = -1$

Sol'n: $\begin{cases} x_1 = -3 \\ x_2 = 4 \\ x_3 = -1. \end{cases}$

#17.(a) $\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is consistent with an ∞ # of sol'ns.
free vars: x_3
leading vars: x_1, x_2, x_4

Backsolve:

$$x_1 + 2x_2 + 2x_4 = 1 \Longrightarrow x_1 = 1 - 2x_2 - 2x_4 = 1 - 2(3 - x_3) \cancel{- 2x_4} = -5 + 2x_3 \cancel{- 2x_4}$$

$$x_2 + x_3 = 3 \Longrightarrow x_2 = 3 - x_3$$

$$x_4 = 0 \Longrightarrow x_4 = 0$$

Sol'n:

$$x_1 = -5 + 2x_3 \cancel{- 2x_4}$$
$$x_2 = 3 - x_3$$
$$x_3 = x_3$$
$$x_4 = 0$$

(b) $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is inconsistent

(c) $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is consistent with a unique solution
leading vars: x_1 & x_2
no free vars.

Backsolve:

$$x_1 + 4x_2 = 2 \Longrightarrow x_1 = 2 - 4x_2 = 2 - 4 = -2$$
$$x_2 = 1 \Longrightarrow x_2 = 1$$

Sol'n: $\begin{cases} x_1 = -2 \\ x_2 = 1 \end{cases}$

(d) $\begin{bmatrix} 2 & 0 & -5 & 1 & 0 & 7 \\ 0 & -1 & 4 & 6 & -3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$ Is consistent w/ an ∞ # of sol'ns.
leading vars: x_1, x_2 & x_4
free vars: x_3 & x_5

Backsolve: $2x_1 \quad -5x_3 + x_4 \quad = 7 \implies x_1 = \frac{1}{2}(7 + 5x_3 - x_4)$

$-x_2 + 4x_3 + 6x_4 - 3x_5 = 0 \implies x_2 = 4x_3 - 6x_4 + 3x_5 = 4x_3 - 6(1+2x_5) + 3x_5 = -6 + 5x_5$

$x_4 - 2x_5 = 1 \implies x_4 = 1 + 2x_5$

$$\begin{aligned} x_1 &= \tfrac{1}{2}(7 + 5x_3 - x_4) \\ &= \tfrac{1}{2}(7 + 5x_3 - (1+2x_5)) \\ &= \tfrac{1}{2}(6 + 5x_3 - 2x_5) \\ &= 3 + \tfrac{5}{2}x_3 - x_5. \end{aligned}$$

Sol'n: $\begin{cases} x_1 = 3 + \frac{5}{2}x_3 - x_5 \\ x_2 = -6 \quad -5x_5 \\ x_3 = \quad x_3 \\ x_4 = 1 \quad + 2x_5 \\ x_5 = \quad x_5. \end{cases}$

(e) $\begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Is consistent w/ an ∞ # of sol'ns.
leading var: x_1
free vars: x_2 & x_3

Backsolve: $2x_1 + x_2 = 2 \implies x_1 = \frac{1}{2}(2 - x_2) = 1 - \frac{1}{2}x_2$

Sol'n: $\begin{cases} x_1 = 1 - \frac{1}{2}x_2 \\ x_2 = \quad x_2 \\ x_3 = \quad x_3 \end{cases}$

#19. (a) False. The echelon form $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ corresponds to a linear system with 3 equations & 2 unknowns that is consistent.

(b) False. If the echelon form is $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ the system has 3 unknowns and 2 equations but is inconsistent.

(c) False. The echelon form $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ corresponds to a homogeneous linear system with a row of zeros that has a unique solution.

(d) False. Every homogeneous system is consistent.

(e) True. For a linear system of 3 equations of 3 unknowns, the fact that the homogeneous system has a unique solution means that all 3 variables are leading variables. In particular, the last row of the matrix in echelon form is not all zeros. Changing the right-hand sides does not change the consistency or the leading variables of the problem. There are no free variables and ~~it~~ the system is consistent. Therefore, the nonhomogeneous system also has a unique solution.