MATH 526 (Section 1 & 2)
Prof. Meade

University of South Carolina
Spring 2007

Exam 3
13 April 2007

Name: Key
SS # (last 4 digits): ____________

Instructions:

1. There are a total of 6 problems (not counting the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. The computers, and Maple, can be used for any part of the exam. In some instances, it will be faster and easier to do hand calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	24	
2	12	
3	30	
4	10	
5	10	
6	14	
Total	100	

Good Luck!

1. (24 points) [8 points each]

(a) Let W be the set of vectors of the form $\begin{bmatrix} a+2b+c \\ 2a-2c \\ -3a+b+4c \end{bmatrix}$.

Show that W is a subspace of $\mathbf{R}^3$ by expressing W as the span of a set of vectors.

$$W = \left\{ \begin{bmatrix} a+2b+c \\ 2a-2c \\ -3a+b+4c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\}$$

(b) Let W be the set of vectors of the form $\begin{bmatrix} a^2 \\ a \end{bmatrix}$. Is W a subspace of $\mathbf{R}^2$?

1. $0 \in W$

2. $\begin{bmatrix} a^2 \\ a \end{bmatrix} + \begin{bmatrix} b^2 \\ b \end{bmatrix} = \begin{bmatrix} a^2+b^2 \\ a+b \end{bmatrix} \neq \begin{bmatrix} (a+b)^2 \\ a+b \end{bmatrix}$ (whenever a & b are non-zero)

so W is <u>not</u> closed under addition. $\therefore$ W is not a subspace of $\mathbb{R}^2$

3. $c\begin{bmatrix} a^2 \\ a \end{bmatrix} = \begin{bmatrix} ca^2 \\ ca \end{bmatrix} \neq \begin{bmatrix} (ca)^2 \\ ca \end{bmatrix}$ (whenever $|c| \neq 1$ & $c \neq 0$)

so W is <u>not</u> closed under scalar multiplication $\therefore$ W is not a subspace of $\mathbb{R}^2$

(c) Let W be the set of matrices of the form $\begin{bmatrix} a & 2 \\ -b & d \end{bmatrix}$. Is W a subspace of M_{22}?

1. $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$ so W is not a subspace of M_{22}

(W is neither closed under addition nor closed under scalar multiplication)

2. (12 points) [6 points each] Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 3 \\ -3 \\ -2 \\ -4 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} -3 \\ 5 \\ -2 \\ 0 \end{bmatrix}$ and define $W = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$

(a) Are the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ linearly dependent or independent? If they are dependent, find a linear dependence relation.

Let $M = [u \ v \ w \ z]$. Using Maple we find the solutions to $Mx = 0$ are in $\text{Span}\left\{\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. This means these 4 vectors are linearly dependent. A linear dependence relation is

$$-2u - v + z = 0.$$

(b) Find a basis for W.

From $-2u - v + z = 0$ we see that z can be omitted without changing the span. Let $N = [u\, v\, w]$. The only solution to $Nx = 0$ is $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so these three vectors are independent.

$\therefore$ a basis for W is $\{u, v, w\}$.

Note: In the same way, $\{u, w, z\}$ and $\{v, w, z\}$ are also bases for W.

3. (30 points) Let $A = \begin{bmatrix} 1 & 1 & -3 & 3 \\ -2 & 0 & 2 & 0 \\ -3 & 2 & -1 & 3 \\ -2 & 1 & 0 & 1 \\ 2 & 0 & -2 & -1 \end{bmatrix}$.

(a) [6 points] Find the echelon form of A.

Using Maple, the echelon form of A is

$$R = \begin{bmatrix} 1 & 1 & -3 & 3 \\ 0 & 2 & -4 & 6 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) [8 points] Find a basis for the column space of A.

Columns 1, 2, & 4 of R have pivot elements,
so columns 1, 2, & 4 of A are a basis for the column space of A.

$$\mathrm{col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(c) [8 points] Find a basis for the null space of A.

The solutions to $Ax = 0$ are $x = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix}$

so $\mathrm{nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) [4 points] What is the dimension of the row space of A?

$\dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)) = 3.$

(e) [4 points] What is the dimension of the null space of A^T?

$$\begin{aligned} \dim(\mathrm{nul}(A^T)) &= \#\text{ col's of } A^T - \dim(\mathrm{col}(A^T)) \\ &= \#\text{ rows of } A - \dim(\mathrm{row}(A)) \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

4. (10 points) Construct a 3×4 matrix A whose column space has dimension 3 and $A \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}$.

- Write A as a matrix in echelon form.
- A must have 3 pivots.
- The extra information tells us that

$$A = \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} = [A_1\, A_2\, A_3\, A_4]$$

$A_1 + 2A_2 - A_4 = 0.$

so $A_4 = A_1 + 2A_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Check: 1. $\dim A = 3.$ ✓

2. $A \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$ ✓

5. (10 points) Use a cofactor expansion to find the determinant of $B = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 0 \\ -3 & 1 & -3 & 2 \end{bmatrix}$.

Show all of your work!

$$\det B = -0 \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ -3 & 1 & -3 \end{bmatrix} + 1 \det\begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & -1 \\ -3 & 1 & -3 \end{bmatrix} - 0 \det\begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} + 2 \det\begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= 1\left(-1 \det\begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix} - (-1) \det\begin{bmatrix} 1 & -1 \\ -3 & -3 \end{bmatrix} + 2 \det\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\right)$$

$$+2\left(-1 \det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} - (-1) \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + 2 \det\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\right)$$

$$= 1\left(-1(-9-(-1)) + 1(-3-3) + 2(1-(-9))\right)$$

$$+2\left(-1(-1-3) + 1(-1-1) + 2(3-1)\right)$$

$$= 1(8 - 6 + 20) + 2(4 + (-2) + 2)$$

$$= 22 + 2(6)$$

$$= 22 + 12$$

$$= 34.$$

6. (14 points) [2 points each] Determine if each statement is true or false. Provide a brief explanation to support each answer.

(a) F If $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a subspace of $\mathbf{R}^3$, then W must have dimension 3.

(b) F If $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\dim W = 3$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for W.

(c) T If the columns of a 5 by 5 matrix span $\mathbf{R}^5$, then the matrix is invertible.

(d) T A matrix and its transpose always have the same rank.

(e) F If the columns of a matrix are linearly dependent, then the rows must also be linearly dependent.

(f) T If A and B are row equivalent matrices, then A and B have the same rank.

(g) T If the row space of a 6 by 4 matrix has dimension 4, then the columns form a linearly independent set of vectors.