MATH 526 (Section 1 & 2)
Prof. Meade

University of South Carolina
Spring 2007

Exam 2
7 March 2007

Name: Key
SS # (last 4 digits): ____

Instructions:

1. There are a total of 6 problems (not counting the Extra Credit problem) on 7 pages. Check that your copy of the exam has all of the problems.
2. The computers, and Maple, can be used for any part of the exam. In some instances, it will be faster and easier to do hand calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	16	
2	10	
3	24	
4	20	
5	14	
6	16	
Extra Credit	10	
Total	100	

Good Luck!

1. (16 points) [8 points each]

   (a) Assume A, B, and C are invertible $n \times n$ matrices. Simplify the following matrix expression as much as possible.
   $$(A^T B^{-1})^{-1}(B^T A)^T.$$

$$= (B^{-1})^{-1}(A^T)^{-1} A^T B^{TT}$$
$$= B\left((A^T)^{-1}A^T\right)B$$
$$= BIB$$
$$= BB$$
$$= B^2$$

   (b) Factor the matrix expression into the product of two matrix expressions.
   $$A^2 - 2AC + 2BA - 4BC.$$

$$= AA - 2AC + 2BA - 4BC$$
$$= A(A-2C) + 2B(A-2C)$$
$$= (A+2B)(A-2C)$$

2. (10 points) Consider the stochastic matrix $P = \begin{bmatrix} 0.8 & 0.5 & 0.8 \\ 0 & 0.5 & 0 \\ 0.2 & 0 & 0.2 \end{bmatrix}$. Find all probability vectors x such that $Px = x$. (Show enough work that I can see how you obtained your answer.)

$$Px = x \iff (P-I)x = 0$$

$$P - I = \begin{bmatrix} -0.2 & 0.5 & 0.8 \\ 0 & -0.5 & 0 \\ 0.2 & 0 & -0.8 \end{bmatrix} \longrightarrow \begin{bmatrix} -0.2 & 0.5 & 0.8 \\ 0 & -0.5 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -0.2 & 0.5 & 0.8 \\ 0 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduce will give you

Makes me give

$x_3 = x_3$
$x_2 = 0$ So $x = \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix}$
$x_1 = 4x_3$

$x_1 + x_2 + x_3 = 5x_3 = 1 \iff x_3 = 4/5$

The probability vector is $x = \begin{bmatrix} 4/5 \\ 0 \\ 1/5 \end{bmatrix}$.

3. (24 points) These problems are intended to be solved by hand.
(Show all steps leading to your answer.)

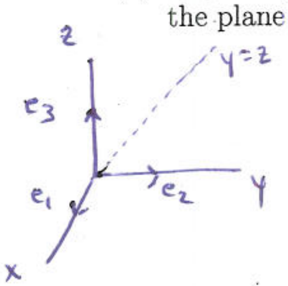
(a) [3 points] Find all values of k for which $B = \begin{bmatrix} 2 & 5 \\ k & 4 \end{bmatrix}$ does not have an inverse.

no inverse when $ad-bc=0$: $2\cdot 4 - k\cdot 5 = 0$

$8-5k=0$

$\underline{k = 8/5}$ (no inverse)

(b) [9 points] Find the 3×3 matrix corresponding to the transformation that reflects across the plane $y=z$.



$Re_1 = e_1$

$Re_2 = e_3$

$Re_3 = e_2$

so $R = \begin{bmatrix} e_1 & e_3 & e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) [12 points] Let $C = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Find the inverse of C.

NOTE: Show all steps involved in finding C^{-1} *by hand.*

$$\begin{bmatrix} C & I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & -2 & 1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \leftrightarrow r_1} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 1 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 2 & -2 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[r_3-2r_1 \to r_3]{r_2+2r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & -2 & 1 & 1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{r_3+r_2 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2} r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= \begin{bmatrix} I & C^{-1} \end{bmatrix}$$

so $C^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

Check: $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1/2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

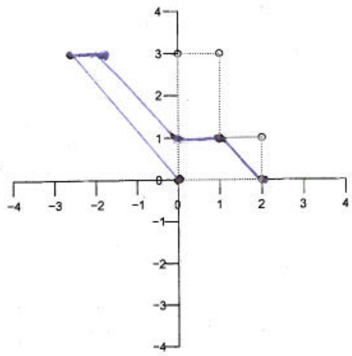
4. (20 points) Here are two matrices of the forms that we have studied.

For each matrix:

- state the type of transformation it performs
- draw the image of the L-shaped figure

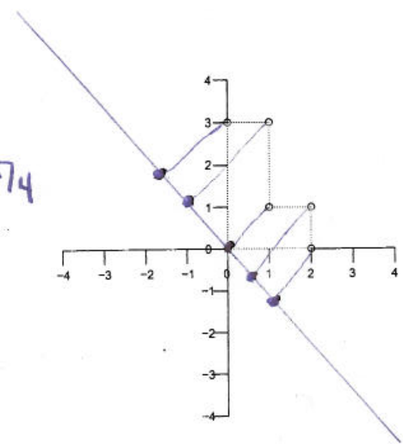
(a) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

horizontal shear $(k=-1)$

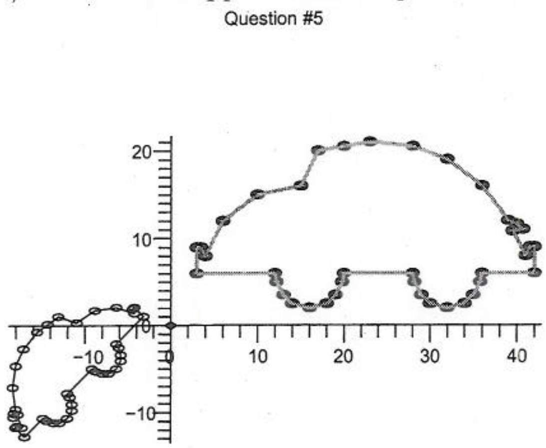


(b) $B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$

projection on line with $\theta = -\pi/4$
i.e. onto the line $y=-x$



5. (14 points) For this problem, refer to the supplemental Maple worksheet.



(a) Identify a sequence of geometric transformations used to transform the thick (red) bug into the thin (black) bug.

NOTE: For example, "Reflect about the x-axis, then project onto the line $y = 3x$."

Here is one possible solution: 1. reflect about the y-axis.

2. rotate $\pi/4$ radians counterclockwise

3. shrink by factor of 2.

Note: These can be done in any order.

Another solution. 1. Reflect about $y=-x$.
2. Rotate $\pi/4$ radians clockwise
3. Shrink by factor of 2.

(Again, these can be done in any order.)

(b) Give the transformation matrix, A, for the composite transformation that maps the thick (red) bug into the thin (black) bug.

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & \sqrt{2}/4 \end{bmatrix}.$$

A := Diagmat([1/2, 1/2]) . Rotatemat(π/4) . Reflectmat(π/2);

6. (16 points) [4 points each] Consider the linear transformations $T(\mathbf{x}) = A\mathbf{x}$ and $S(\mathbf{x}) = B\mathbf{x}$ where $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix}$. $T: \mathbb{R}^3 \to \mathbb{R}^2$ $S: \mathbb{R}^3 \to \mathbb{R}^3$

(a) Find the kernel of T. solve $Ax = 0$ $\begin{bmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{\text{Reduce}} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 4 & 2 \end{bmatrix}$

Matsolve (A, zero) returns $\begin{bmatrix} 3/2\, x_3 \\ -1/2\, x_3 \\ x_3 \end{bmatrix}$ so the kernel of T is the span of $\begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix}$.

(b) Find the range of S. when is $Bx = b$ solvable?

$$\begin{bmatrix} 3 & 1 & -2 & b_1 \\ 2 & 2 & -2 & b_2 \\ 1 & -1 & 0 & b_3 \end{bmatrix} \xrightarrow{\text{Reduce}} \begin{bmatrix} 3 & 1 & -2 & b_1 \\ 0 & 4/3 & -2/3 & b_2 - \frac{2}{3} b_1 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{bmatrix}$$

The range of S is all vectors $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $b_3 - b_1 + b_2 = 0$.

(c) Which of the following compositions are defined: $S \circ S$, $S \circ T$, $T \circ S$ and $T \circ T$? (Explain.)

$S \circ S$ is defined because $S(x) \in \mathbb{R}^3$ and so $S(S(x))$ is defined.
$S \circ T$ is not defined because $T(x) \in \mathbb{R}^2$ and $S(T(x))$ is not defined.
$T \circ S$ is defined because $S(x) \in \mathbb{R}^3$ and so $T(S(x))$ is defined.
$T \circ T$ is not defined because $T(x) \in \mathbb{R}^2$ and $T(T(x))$ is not defined.

(d) Find each of the defined compositions in (c), find the corresponding matrix for that transformation.

$(S \circ S)(x) = S(S(x)) = S(Bx) = B(Bx) = (B^2)x$ where

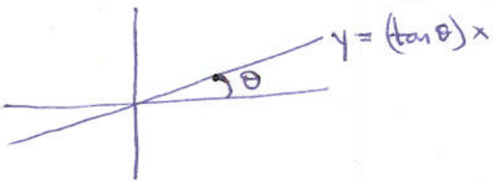
$$B^2 = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 7 & -8 \\ 8 & 8 & -8 \\ 1 & -1 & 0 \end{bmatrix}$$

$(T \circ S)(x) = T(S(x)) = T(Bx) = A(Bx) = (AB)x$ where

$$AB = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 2 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 8 & 8 & -8 \end{bmatrix}.$$

Extra Credit (10 points)Let P be the matrix transformation from R^2 to R^2 that projects vectors onto the line through the origin making an angle of θ radians with the x-axis. Let $\mathbf{u}$ be a unit vector lying along this line and $\mathbf{v}$ be a unit vector perpendicular to this line.

(a) Explain why $\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.



when $x = \cos\theta$

$y = \tan\theta \cos\theta = \sin\theta.$

So $u = \alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ & choosing $\alpha = 1$ makes $\|u\| = 1$.

$v \cdot u = 0$ & also has $\|v\| = 1$ so $v \perp u$ & also unit length

(b) What is the value of $P\mathbf{u}$?

$Pu = u$

(c) What is the value of $P\mathbf{v}$?

$Pv = 0$

(d) Combine the information in (b) and (c) into a matrix equation $PA = B$.

$P[u \;\; v] = [u \;\; 0]$ so $A = [u \;\; v] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ✓ rotation by θ. (invertible)

$B = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$

(e) Solve the equation in (d) for the matrix P.

$PA = B \iff PAA^{-1} = BA^{-1}$

$P(I) = BA^{-1}$

$P = BA^{-1}$

$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ rotation by $-\theta$.

So

$P = BA^{-1} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}.$