MATH 526 (Section 1 & 2)
Prof. Meade

University of South Carolina
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Exam 1
7 February 2007

Name: Key
SS # (last 4 digits): ____________

Instructions:

1. There are a total of 6 problems (not counting the Extra Credit problem) on 8 pages. Check that your copy of the exam has all of the problems.
2. The computers, and Maple, can be used for any part of the exam. In some instances, it will be faster and easier to do hand calculations.
3. Be sure you answer the questions that are asked.
4. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
5. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
6. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	18	
2	18	
3	18	
4	15	
5	16	
6	15	
Extra Credit	10	
Total	100	

Good Luck!

1. (18 points) [6 points each] Consider the system of linear equations:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 4x_2 & + & 3x_3 & = & 1 \\ -3x_2 & - & 7x_2 & + & 2x_3 & = & 2 \end{array}$$

(a) Write the augmented matrix for this system.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ -3 & -7 & 2 & 2 \end{bmatrix}$$

(b) Show all row operations needed to reduce the augmented matrix to echelon form.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ -3 & -7 & 2 & 2 \end{bmatrix} \xrightarrow[r_3+3r_1 \to r_3]{r_2-2r_1 \to r_2} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2 \longrightarrow r_3} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(c) Use back substitution to find all solutions to the system.

$x_1 + 2x_2 + x_3 = 0 \Longrightarrow x_1 = -2x_2 - x_3 = -6-1=-$

$-x_2 + 5x_3 = 2 \Longrightarrow x_2 = 5x_3 - 2 = 5-2 = 3$

$x_3 = 1 \Longrightarrow x_3 = 1$

The solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$.

2. (18 points) [9 points each] For each of the following two systems of linear equations:

(i) find the echelon form for the augmented matrix (3 points),

(ii) determine if the system is consistent or inconsistent (3 points),

(iii) if the system is consistent, find all solutions.

or if the system is inconsistent. explain how you know it is inconsistent (3 points).

(a)

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 6x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & 9x_2 & + & x_3 & = & 6 \end{array}$$

Augmented matrix: (M3) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 3 & 4 \\ 3 & 9 & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This system is consistent. (with one free variable: x_2)

The solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-3x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

(b)

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 6x_2 & + & 2x_3 & = & 5 \\ 3x_1 & + & 9x_2 & + & 3x_3 & = & 6 \end{array}$$

Augmented matrix: (M5) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 2 & 5 \\ 3 & 9 & 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This system is inconsistent.

The second row says $0x_1 + 0x_2 + 0x_3 = 1$ which is impossible.

3. (18 points) The following linear system of equations involves two parameters, h and k.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & = & k \\ x_1 & + & 3x_2 & + & hx_3 & = & 5 \end{array}$$

(a) [6 points] Find the echelon form for the augmented matrix.

Augmented matrix: (N1) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & k \\ 1 & 3 & h & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & k-1 \\ 0 & 0 & h-3 & 6-2k \end{bmatrix}$

(b) [4 points] For what values of h and k does the system have no solutions?

There are no solutions when $h-3=0$ and $6-2k \neq 0$.
$(h=3)$ $(k \neq 3)$.

(c) [4 points] For what values of h and k does the system have exactly one solution?

There is exactly one solution when $h-3 \neq 0$.
$(h \neq 3)$

(d) [4 points] For what values of h and k does the system have an infinite number of solutions?

There are an infinite number of solutions when
$h-3=0$ and $6-2k=0$
$(h=3)$ $(k=3)$.

4. (15 points) [5 points each] Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 6 \\ 2 \\ 2 \\ 4 \\ -2 \end{bmatrix}$.

(a) Find the general solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$.

$$\begin{bmatrix} 0 & 3 & 3 & 6 & 0 \\ 3 & 1 & 4 & 2 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 1 & -1 & 0 & -2 & 0 \end{bmatrix} \xrightarrow[(H10)]{} \begin{bmatrix} 3 & 1 & 4 & 2 & 0 \\ 0 & 3 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

for any values of c_1 & c_2.

(b) Find a linear dependence relation that involves exactly two of the four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

Choose $c_3 = 0$ & $c_4 = 1$ to find the solution

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

This gives the linear dependence relation:

$$-2v_2 \quad + v_4 = 0.$$

(c) Find a linear dependence relation that involves all four vectors.

To have all 4 coefficients not zero requires:

$$\begin{aligned} -c_3 &\neq 0 \\ -c_3 - 2c_4 &\neq 0 \\ c_3 &\neq 0 \\ c_4 &\neq 0. \end{aligned}$$

A simple choice for c_3 & c_4 is $c_3 = c_4 = 1$. Then $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

and a linear dependence relation is:

$$-v_1 - 3v_2 + v_3 + v_4 = 0.$$

5. (16 points) [4 points each] Let $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} -3 \\ 1 \\ -4 \\ -4 \end{bmatrix}$, and $\mathbf{w}_3 = \begin{bmatrix} 4 \\ -1 \\ -1 \\ 2 \end{bmatrix}$. We want to find out if the set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent or dependent.

(a) Write the vector equation that must be solved.

$c_1 w_1 + c_2 w_2 + c_3 w_3 = 0.$

(b) Write the vector equation as a linear system.

$$\begin{aligned} -c_1 - 3c_2 + 4c_3 &= 0 \\ 2c_1 + c_2 - c_3 &= 0 \\ 3c_1 - 4c_2 - c_3 &= 0 \\ c_1 - 4c_2 + 2c_3 &= 0 \end{aligned}$$

(c) Find all solutions to this system.

Augmented matrix: (M9)

$$\begin{bmatrix} -1 & -3 & 4 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & -4 & -1 & 0 \\ 1 & -4 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -3 & 4 & 0 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & -\frac{36}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Has only the trivial solution: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

(d) Is the set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent or independent?

These vectors are linearly independent (because there is <u>no</u> linear dependence relation).

6. (15 points) This problem asks about finding a curve with specific properties. In each case. (i) clearly identify the system of linear equations that must be solved, (ii) list the the echelon form of the corresponding augmented matrix, and (iii) answer the question.

(a) [3 points] What is the general form of a polynomial of degree 2?

$$p(x) = ax^2 + bx + c.$$

(b) [6 points] Find all polynomials of degree 2 whose graph goes through the points $(1, 3)$ and $(3, 6)$.

$p(1) = 3$: $a + b + c = 3$

$p(3) = 6$: $9a + 3b + c = 6$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 9 & 3 & 1 & 6 \end{bmatrix} \xrightarrow[(\text{H7})]{} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -6 & -8 & -21 \end{bmatrix}$$

Solutions: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1/2 + 1/3\,c \\ 7/2 - 4/3\,c \\ c \end{bmatrix}$.

Polynomials: $p(x) = \left(-\frac{1}{2} + \frac{c}{3}\right)x^2 + \left(\frac{7}{2} - \frac{4}{3}c\right)x + c$

(c) [6 points] Find all polynomials of degree 2 whose graph goes through the points $(1, 3)$ and $(3, 6)$ and has slope -1 at $x = 2$.

$p'(x) = 2ax + b$

$p(1) = 3$: $a + b + c = 3$

$p(3) = 6$: $9a + 3b + c = 6$

$p'(2) = -1$: $4a + b = -1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 9 & 3 & 1 & 6 \\ 4 & 1 & 0 & -1 \end{bmatrix} \xrightarrow[(\text{H6})]{} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -6 & -8 & -21 \\ 0 & 0 & 0 & -5/2 \end{bmatrix}$$

This system is inconsistent.

There are no quadratic polynomials with these three properties.

Extra Credit (10 points)Prove the following statement: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbf{R}^n$ that includes the zero vector, then the set is linearly dependent.

HINT: Assume $\mathbf{v}_1$ is the zero vector.

To show $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent we have to demonstrate that there is a linear dependence relation for these vectors.

Assuming $v_1 = \vec{0}$, then

$$\cancel{0\vec{v}_1 +}$$

$$1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + \ldots + 0\vec{v}_k = \vec{0}$$

is a linear dependence relation ($c_1 = 1 \neq 0$, $c_2 = \ldots = c_k = 0$).

Thus, $\{v_1, v_2, \ldots, v_k\}$ is a linearly dependent set of vectors.