Eigenvalue Analysis of a Model for an Owl Population[1]

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Overview

This final project uses eigenvalues and eigenvectors to study the change in a population over time.

The only new MATLAB command introduced in this lab is `abs`, the absolute value of a real number or the magnitude of a complex number.

Description of the Original Model

The Lamberson model[2] for a spotted owl population divides the total population into three classes based on age: juvenile (up to 1 year old), subadult (1 to 2 years old), and adult (over 2 years old). The population is examined at yearly intervals. Since it is assumed that the number of male and female owls is equal, only female owls are counted in the analysis. In this model only adult females produce offspring (at the rate of 0.33 juvenile females per year). The survival rates are 18% for the juvenile females (who enter their second year of life as subadults), 71% for the subadults (who return for their third year of life as an adult), and 94% of the adult females survive each year. All of these rates are assumed to remain constant through time.

Suppose, in year k, there are j_k juvenile females, s_k subadult females, and a_k females. Lamberson's model for the populations in the subsequent year $(k+1)$ uses

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

where $\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$. (To be sure you understand this model, write out the equations and see how every term fits with the description of the model given above.)

The goal is to determine the long-term dynamics of the population: does the population become extinct? does it approach a sustainable equilibrium? or does the population grow? How does this outcome depend on the initial distribution of owls?

The Role of the Dominant Eigenvalue

To understand how the eigenvalues of $\mathbf{A}$ can be used to analyze this model, it must be noted that the initial distribution of owls can be expressed in the form

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are the three eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues λ_1, λ_2, and λ_3. (For convenience, assume that λ_1 is the dominant eigenvalue of $\mathbf{A}$.) Then,

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 \\ &= \mathbf{A}\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\right) \\ &= c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 + c_3\mathbf{A}\mathbf{v}_3 \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3 \end{aligned}$$

and, in general,

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k \\ &= c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3. \end{aligned}$$

Notice that if $|\lambda_1| < 1$ then the population becomes extinct, if $|\lambda_1| > 1$ then the population grows without bound. Only when $|\lambda_1| = 1$ does the population reach a sustainable equilibrium.

[1]Based on a Case Study in *Linear Algebra and Its Applications*, Third Edition, by David C. Lay. The URL for the specific Case Study is `http://media.pearsoncmg.com/aw/aw_lay_linearalg_updated_cw_3/cs_apps/lay03_05_cs.pdf`.

[2] Lamberson et al., "A Dynamic Analysis of the Viability of the Northern Spotted Owl in a Fragmented Forest Environment", *Conservation Biology* **6** (1992), pp. 505–512.

The following MATLAB commands identify the dominant eigenvalue of $\mathbf{A}$.

```
≫ [V D] = eig( A )            % eigenvalues and eigenvectors of A
≫ abs( diag(D) )
≫ lambda1 = D(3,3)            % identify dominant eigenvalue
≫ lambda2 = D(1,1)            % other eigenvalues
≫ lambda3 = D(2,2)
```

What is the dominant eigenvalue of $\mathbf{A}$? Verify that $|\lambda_1| < 1$ and $|\lambda_2| = |\lambda_3| < |\lambda_1|$. Explain why this means the owl population will become extinct.

Interpretation of the Dominant Eigenvector

The components of the eigenvector $\mathbf{v}_1$ corresponding to the dominant eigenvalue λ_1, i.e., the dominant eigenvector, provide additional information about the long-term distributions of the owl subpopulations. Recall that any nonzero multiple of an eigenvector is also an eigenvector for the same eigenvalue. Thus, it is possible to assume $\mathbf{v}_1$ is rescaled so that $\|\mathbf{v}_1\|_1 = 1$, i.e., the sum of the three components in $\mathbf{v}_1$ is 1. The components of this normalized eigenvector give the relative size of the juvenile, subadult, and adult subpopulations, respectively.

Find the vector $\mathbf{v}_1$ with $\|\mathbf{v}_1\|_1 = 1$. After many years, what fraction of the owl population is juveniles? subadults? and adults?

The following MATLAB commands identify the normalized dominant eigenvector of $\mathbf{A}$.

```
≫ v1 = V(:,3)                 % dominant eigenvector
≫ v1 = v1/norm(v1,1)          % normalized dominant eigenvector
```

Simulation for a Specific Initial Distribution

Suppose the spotted owl population is 300 with 100 owls in each subpopulation. Find values for c_1, c_2, and c_3 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ where the three eigenvectors are rescaled so that $\|\mathbf{v}_1\|_1 = \|\mathbf{v}_2\|_1 = \|\mathbf{v}_3\|_1 = 1$.

The following MATLAB commands compute the estimated owl populations for a ten year period.

```
≫ X(:,1)=x0                   % the initial distribution
≫ for k=1:10,
≫   X(:,k+1) = A*X(:,k);      % x_{k+1} = A x_k
≫ end
≫ X                           % display all iterates
```

Do these results confirm the results of the eigenanalysis? In particular, is the population heading towards extinction? How long would it take for the population to decay to a total of 10 owls? 1 owl? How long does it take for the relative sizes of the three subpopulations to be close to the values in the normalized dominant eigenvector?

Clear all variables before you begin to work on Part II.