Condition Numbers and Relative Errors

Susanne Brenner and Li-Yeng Sung
(modified by Douglas B. Meade)
Department of Mathematics

Overview

It does not make any sense to make a statement such as $\mathbf{v} > 0$ or $\mathbf{A} \le 1$. To talk about inequalities it is necessary to compare scalar quantities. The Euclidean norm of a vector ($\|\mathbf{v}\| = \sqrt{\mathbf{v}^t\mathbf{v}}$) is the standard "norm" for a vector. There are many other norms for both vectors and matrices. The condition number is a scalar quantity that provides useful information about a matrix.

In this lab, norms are used to measure different errors associated with the numerical solution of $\mathbf{Ax} = \mathbf{b}$. You will see that there can be a significant difference between the relative error between the exact solution $\mathbf{x}_{ex}$ and the computed (approximate) solution $\mathbf{x}_{ap}$ ($\frac{\|\mathbf{x}_{ex}-\mathbf{x}_{ap}\|}{\|\mathbf{x}_{ex}\|}$) and the relative error of the right-hand sides ($\frac{\|\mathbf{b}-\mathbf{A}\mathbf{x}_{ap}\|}{\|\mathbf{b}\|}$).

New `MATLAB` commands introduced in this lab include `norm`, `cond`, and `inv`.

Part I

Let $\mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ 5 \\ -1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 & 5 \\ -3 & 2 & 1 & -4 \\ 5 & -6 & -1 & 0 \\ 2 & 5 & 0 & 12 \end{bmatrix}$.

<u>Vector Norms</u>

```
≫ norm( x, 1 )          % vector 1-norm of x
≫ norm( x, 2 )          % vector 2-norm of x
≫ norm( x )             % default norm for vectors is the 2-norm
≫ norm( x, inf )        % vector ∞-norm of x
```

<u>Matrix Norms</u>

```
≫ norm( A, 1 )          % matrix 1-norm of A
≫ norm( A, 2 )          % matrix 2-norm of A
≫ norm( A )             % default matrix norm is the 2-norm
≫ norm( A, inf )        % matrix ∞-norm of A
```

<u>The Inverse of a Matrix</u>

```
≫ inv( A )              % the inverse of A
```

<u>The Condition Number of a Matrix</u>

The condition number, $\kappa(\mathbf{A})$, of a nonsingular $n \times n$ matrix $\mathbf{A}$ is defined to be $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$. Note that using different matrix norms gives different condition numbers.

```
≫ norm( A ) * norm( inv(A) )   % condition number of A using the default norm (2)
≫ cond( A )                    % condition number of A using the default norm (2)
≫ cond( A, 1 )                 % condition number of A using the 1-norm
≫ cond( A, inf )               % condition number of A using the ∞-norm
```

Clear all variables before you begin to work on Part II.