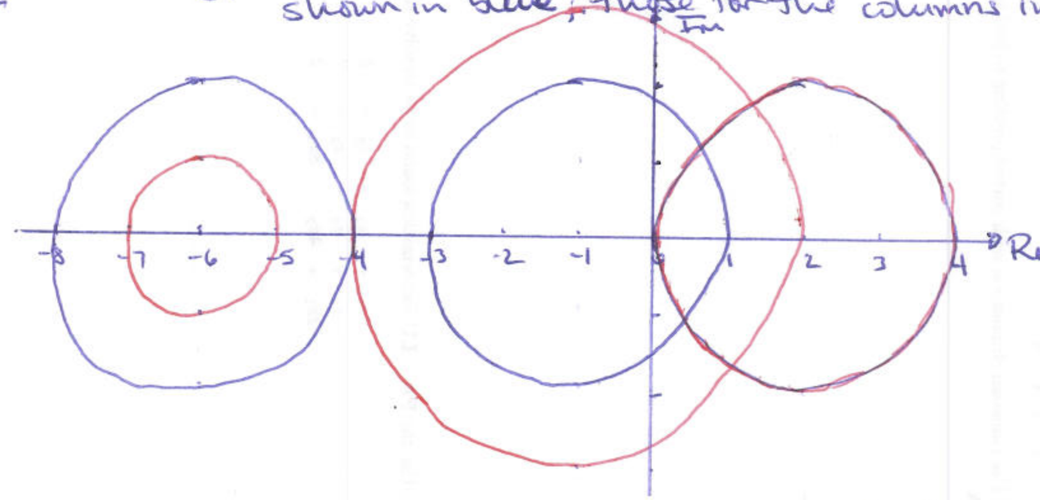
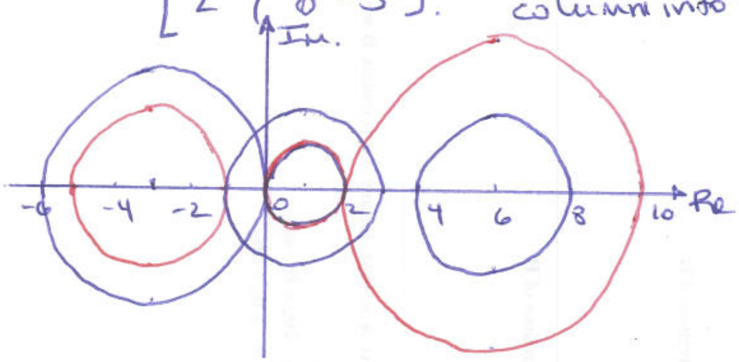
# HW #12 – Solutions

#14.7. $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -6 \end{bmatrix}$. The sketch for the Gershgorin Circle Theorem is shown below. The disks from the rows are shown in blue, those for the columns in red.




We can conclude that there is one eigenvalue in the interval $[-7, -5]$ and the other two eigenvalues are inside the circle w/ radius 2 centered at ~~x~~ $(-1,0)$ & the circle w/ radius 2 centered at $(2,0)$. Because these 2 e-values have an absolute value (modulus) that cannot exceed 4, we see that the eigenvalue in $[-7,-5]$ has a larger absolute value & will be the dominant eigenvalue for this matrix.

#14.8 $A = \begin{bmatrix} 6 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & -3 \end{bmatrix}$. The sketch for the Gershgorin Circle Theorem is shown below. Again: row info is in blue & column info in red.




From the row information:
1 e-value in $[4, 8]$
& 3 e-values in the union of the circles w/ radius 2 centered at $(1,0)$ and w/ radius 3 centered at $(-3,0)$.

From the column information there is 1 e-value in $[-5,-1]$, 2 e-values in the circle w/ radius 1 & center $(1,0)$, and 1 e-value in $[2,10]$.

Combined, we conclude there is one e-value in $[-5,-1]$, 2 e-values in the circle w/ radius 1 & center $(1,0)$, and 1 e-value in $[4,8]$.

#14.10. To find the eigenvalue of $A = \begin{bmatrix} 6 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 1 & 4 \end{bmatrix}$ inside the unit circle centered at (1,0), use the Shifted Inverse Power Method with $s = 1$.

#15.1. Claim: If one of the vectors in the collection $\{v_1, v_2, \dots, v_n\}$ is the zero vector, then the collection of vectors is linearly dependent.

Proof 1: Suppose the vector $v_j = 0$ for some $j = 1, 2, \dots n$.
Then, the linear combination
$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$
has a non-trivial solution $c_j = 1$ & all other $c_i = 0$.

Proof 2: If $m = n$, then we could form the matrix $A$ with the vectors $v_1, v_2, \dots, v_n$ as its columns. Then, because one of the columns of $A$ is the zero vector we would know that $\det A = 0$, i.e. the vectors $\{v_1, v_2, \dots, v_n\}$ are linearly dependent.

#15.2. (a) $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Because $m = n = 2$ we can form $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ and compute $\det A = 1 + 6 = 7 \neq 0$ to see that $\{v_1, v_2\}$ is a linearly independent collection of vectors in $\mathbb{R}^2$.
(Also, it's obvious that $v_1 \neq \alpha v_2$.)

(b) $v_1 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -3 \\ -9 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$. Again, $m = n = 3$, so form $A = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -3 & -3 \\ 5 & -9 & -5 \end{bmatrix}$.

Then $\det A = (-2) \det \begin{bmatrix} -3 & -3 \\ -9 & -5 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & -3 \\ 5 & -5 \end{bmatrix} + (1) \det \begin{bmatrix} 1 & -3 \\ 5 & -9 \end{bmatrix}$
$$= -2(15 - 27) - 3(-5 + 15) + (-9 + 15) = 24 - 30 + 6 = 0$$
so that this collection of vectors is linearly dependent in $\mathbb{R}^3$.

#15.3. $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ has eigenpairs $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3$, $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Because $v_1 \neq \alpha v_2$ it is clear that $\{v_1, v_2\}$ is a linearly independent collection of vectors. (Also $V = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ has $\det V = 2-1 = 1 \neq 0$.)

#15.7. $v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Gram-Schmidt Orthogonalization

yields: $u_1 = v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$$u_2 = v_2 - \frac{v_2^t u_1}{u_1^t u_1} u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \\ -1 \end{bmatrix}.$$

$$u_3 = v_3 - \frac{v_3^t u_1}{u_1^t u_1} u_1 - \frac{v_3^t u_2}{u_2^t u_2} u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{5}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{-3/5}{9/5}\begin{bmatrix} 4/5 \\ 2/5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Check: $u_1^t u_2 = \frac{4}{5} - \frac{4}{5} = 0$, $u_1^t u_3 = \frac{2}{3} - \frac{2}{3} = 0$, $u_2^t u_3 = \frac{8}{15} + \frac{2}{15} - \frac{2}{3} = 0$. ✓

#15.9. To normalize the orthogonal vectors found in #15.7

$$z_1 = \frac{1}{\|u_1\|_2} u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$z_2 = \frac{1}{\|u_2\|_2} u_2 = \frac{1}{3/\sqrt{5}}\begin{bmatrix} 4/5 \\ 2/5 \\ -1 \end{bmatrix} = \frac{\sqrt{5}}{3}\begin{bmatrix} 4/5 \\ 2/5 \\ -1 \end{bmatrix} = \frac{\sqrt{5}}{3 \cdot 5}\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix} = \frac{1}{3\sqrt{5}}\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}.$$

$$z_3 = \frac{1}{\|u_3\|_2} u_3 = \frac{1}{1}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

For #15.6: $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

G-S orthog. yields: $u_1 = v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$u_2 = v_2 - \frac{v_2^t u_1}{u_1^t u_1} u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

Normalization yields: $z_1 = \frac{1}{\|u_1\|_2} u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$z_2 = \frac{1}{\|u_2\|_2} u_2 = \frac{1}{3/\sqrt{2}}\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \frac{\sqrt{2}}{3} \cdot \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$