#14.3 $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ $P(\lambda) = \det(A - \lambda I) = \cancel{\det(A-\lambda I)} = \det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$

$= (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 1$.

#14.4. $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ $P(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$

$= (2-\lambda)^2 - 9 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$. dominant eigenvalue.

With $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $z_0 = \dfrac{x_0}{\|x_0\|_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_1 = A z_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\alpha_0 = z_0^T x_1 = 2$.

$z_1 = \dfrac{x_1}{\|x_1\|_2} = \begin{bmatrix} 0.8321 \\ 0.5547 \end{bmatrix}$, $x_2 = A z_1 = \begin{bmatrix} 3.3282 \\ 3.6056 \end{bmatrix}$, $\alpha_1 = z_1^T x_2 = 4.7692$

$z_2 = \dfrac{x_2}{\|x_2\|_2} = \begin{bmatrix} 0.6783 \\ 0.7348 \end{bmatrix}$, $x_3 = A z_2 = \begin{bmatrix} 3.6610 \\ 3.5044 \end{bmatrix}$, $\alpha_2 = z_2^T x_3 = 4.9904$.

Yes, it appears that the α_i's are converging to the dominant eigenvalue, $\lambda_1 = 5$.

The relative error of α_2 is $\dfrac{|\alpha_2 - \lambda_1|}{|\lambda_1|} = \dfrac{0.0096}{5} = 0.00192$.

#14.5. The eigenvectors for $\lambda_1 = 5$ are $v_1 = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The eigenvectors for $\lambda_2 = -1$ are $v_2 = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The power method is guaranteed to converge when $x_0 = c_1 v_1 + c_2 v_2$ with $c_1 \neq 0$. When $x_0 = 0 v_1 + c_2 v_2$ the Power Method is not known to converge. In fact, with $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $z_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$, $x_1 = A z_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} / \sqrt{2}$, $\alpha_0 = -1$, $z_1 = x_1$, $x_2 = A z_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$, $\alpha_1 = -1$, ... Every $\alpha_i = -1$.

#14.6. $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$.

With $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: $z_0 = \dfrac{x_0}{\|x_0\|_2} = \begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}$, $x_1 = A z_0 = \begin{bmatrix} 2.3094 \\ 2.3094 \\ 1.1547 \end{bmatrix}$, $\alpha_0 = z_0^T x_1 = 3.3333$

$z_1 = \dfrac{x_1}{\|x_1\|_2} = \begin{bmatrix} 0.6667 \\ 0.6667 \\ 0.3333 \end{bmatrix}$, $x_2 = A z_1 = \begin{bmatrix} 1.3333 \\ 3.0000 \\ 1.6667 \end{bmatrix}$, $\alpha_1 = z_1^T x_2 = \cancel{2.8279}$ 3.4444

$z_2 = \dfrac{x_2}{\|x_2\|_2} = \begin{bmatrix} 0.3681 \\ 0.8148 \\ 0.4527 \end{bmatrix}$, $x_3 = A z_2 = \begin{bmatrix} 1.3580 \\ 2.2634 \\ 1.0864 \end{bmatrix}$, $\alpha_2 = z_2^T x_3 = 2.8279$

HW #11 - Solutions

#13.9. $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

For Jacobi: $S = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$, so $S^{-1} = \begin{bmatrix} 1/2 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

and $S^{-1}T = \begin{bmatrix} 0 & -1/2 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}$. The eigenvalues of $S^{-1}T$

are $\lambda_1 \approx -1.4757$ and $\lambda_{2,3} \approx 0.7378 \pm 0.3650i$. Thus,

$\rho(S^{-1}T) = \max\{1.4757, 0.8232\} = 1.4757 > 1$.

This means Jacobi's method diverges.

For Gauss-Seidel: $S = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $T = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $S^{-1}T = \begin{bmatrix} 0 & -1/2 & -1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Because $S^{-1}T$ is upper triangular, its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1/2$, $\lambda_3 = 0$ and $\rho(S^{-1}T) = \max\{0, 1/2\} = 1/2 < 1$.

This means Gauss-Seidel method converges.

#11. Claim: if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n for $n = 0, 1, 2, \dots$

Proof: Let λ be an eigenvalue of A (with eigenvector v). Then, $Av = \lambda v$.

$$\begin{aligned} \text{Then } A^n v &= A^{n-1}(Av) = A^{n-1}(\lambda v) = \lambda A^{n-1} v \\ &= \lambda A^{n-2}(Av) = \lambda A^{n-2}(\lambda v) = \lambda^2 A^{n-2} v \\ &\qquad\qquad = \lambda^{n-1}(Av) \\ &= \cdots \\ &= \lambda^{n-1}(\lambda v) = \lambda^n v. \end{aligned}$$

That is, λ^n is an eigenvalue of A^n. ∎

#15. (a) $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1, \lambda_2 = 3, \lambda_3 = -1$ so $\rho(A) = \max\{1, 3\} = 3$.

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \lambda_{1,2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_3 = 1$ so $\rho(A) = \max\{1, 1\} = 1$.

(c) $A = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 3, \lambda_4 = 1$ so $\rho(A) = \max\{2, 1, 3\} = 3$.