HW #10 - Solutions

#12.4 (a) $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix}$

$$\det A = \det \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix} \quad \text{interchange 2 columns}$$

$$= \det \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \quad ②+4①\to②$$

$$= 1 \cdot -5 = -5$$

Note: There are many ways to use the properties of the determinant to find these values. I do not expect you to work each problem exactly as I do.

(c) $A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ -1 & 0 & -2 & 0 \\ 3 & 2 & 0 & -1 \end{bmatrix}$

$$\det A = \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ -1 & 0 & -2 & 0 \\ 3 & 2 & 0 & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & -3 & 3 \\ 0 & 2 & 3 & -10 \end{bmatrix} \quad \begin{matrix} ③+①\to③ \\ ④-3①\to④ \end{matrix}$$

$$= \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 4 & -8 \end{bmatrix} \quad ④+②\to④$$

$$= (-3)(4) \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \left\{ \begin{matrix} \text{factor out} \\ -3 \text{ from } ③ \\ 4 \text{ from } ④ \end{matrix} \right.$$

$$= (-3)(4) \det \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad ④-③\to④$$

$$= (-3)(4) \cdot (1)(-2)(1)(-1)$$

$$= -12 \cdot 2$$

$$= -24.$$

#12.6 (a) $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

expand along 2nd column

$$\det A = (-1)^{1+2}(-1)\det\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} + 0 + 0$$
$$= (-1)(-1)\left(1\cdot 3 - 1\cdot(-1)\right)$$
$$= 3+1$$
$$= 4$$

(b) $A = \begin{bmatrix} 0 & 2 & 0 & -3 \\ 1 & -1 & 4 & 9 \\ 0 & 1 & 0 & 0 \\ 5 & 3 & -2 & 7 \end{bmatrix}$ ← expand along 3rd row.

$$\det A = 0 + (-1)^{3+2}(1)\det\begin{bmatrix} 0 & 0 & -3 \\ 1 & 4 & 9 \\ 5 & -2 & 7 \end{bmatrix} + 0 + 0$$ ← expand along 1st row

~~= - det~~

$$= -\left(0 + 0 + (-1)^{1+3}(-3)\det\begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}\right)$$
$$= -\left(-3(-2-20)\right)$$
$$= 3(-22)$$
$$= -66.$$

#12.7. A matrix is singular iff its determinant is 0.

None of the matrices in #12.2 and #12.6 have a determinant that is zero, so none of these matrices is singular, all are nonsingular.

#12.8. When A is an $n\times n$ nonsingular matrix, $\det A \neq 0$ and A^{-1} exists.

Because $AA^{-1} = I$ we know that

$$\det(AA^{-1}) = \det I$$
$$(\det A)(\det(A^{-1})) = 1$$
$$\det(A^{-1}) = \frac{1}{\det A}.$$

#13.1 (a) $A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ $\quad p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{bmatrix}$

$$= (2-\lambda)(2-\lambda) + 3$$
$$= \lambda^2 - 2\lambda + 2\lambda - 4 + 3$$
$$= \lambda^2 - 1$$
$$= (\lambda+1)(\lambda-1) \quad \text{so } \lambda_1 = -1 \;\&\; \lambda_2 = 1.$$

To find the eigenvectors:

$\lambda_1 = -1: \; A - \lambda_1 I = A + I = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \xrightarrow{① + ② \to ②} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ so $3x_1 + x_2 = 0.$

Let x_2 be the free variable, then $3x_1 = -x_2$, $x_1 = -\frac{1}{3}x_2$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \end{bmatrix} = -\frac{1}{3}x_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$

Choose $v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ (or any non-zero multiple of this vector)

$\lambda_2 = 1: \; A - \lambda_2 I = A - I = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \xrightarrow{② + 3① \to ②} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so $x_1 + x_2 = 0$

Let x_2 be the free variable, then $x_1 = -x_2$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

Choose $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

#13.2 (b) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad p(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1$
$$= \lambda^2 - 2\lambda + 1 - 1$$
$$= \lambda^2 - 2\lambda$$
$$= \lambda(\lambda - 2)$$

$\lambda_1 = 0: \; A - \lambda_1 I = A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{② + ① \to ②} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $x_1 - x_2 = 0.$ $\quad x_1 = x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2: \; A - \lambda_2 I = A - 2I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{② - ① \to ②} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$ so $-x_1 - x_2 = 0$, $\quad x_1 = -x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so that $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

#13.3. Let A be a triangular matrix, then A is either upper-triangular or lower triangular. Let's assume that A is upper-triangular, then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \quad \text{and} \quad A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & & a_{1n} \\ 0 & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{nn}-\lambda \end{bmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

(Note that we would have the same result if A is lower-triangular.)

Because $p(\lambda)=0$ for $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}, \ldots, \lambda_n = a_{nn}$

these diagonal elements are the eigenvalues of A.

#13.5 (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$ $\quad A-\lambda I = \begin{bmatrix} 1-\lambda & 1 & 2 \\ 0 & 1-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{bmatrix}$

expand along 1st column.

$$p(\lambda) = \det(A-\lambda I) = (1-\lambda)\det\begin{bmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix} = (1-\lambda)\left((1-\lambda)^2-4\right)$$

$$= (1-\lambda)(\lambda^2-2\lambda+1-4) = (1-\lambda)(\lambda^2-2\lambda-3) = (1-\lambda)(\lambda-3)(\lambda+1)$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -1$.

(b) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad A-\lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} \begin{matrix} -\lambda & 1 \\ 0 & -\lambda \\ 1 & 0 \end{matrix}$

$$\det(A-\lambda I) = (-\lambda)^3 + 1 + 0 - (0+0+0) = -\lambda^3+1 = (\lambda-1)(-\lambda^2-\lambda-1)$$
$$= (1-\lambda)(\lambda^2+\lambda+1)$$

so the eigenvalues are $\lambda_1 = 1$, and (by the Quadratic Formula) $\lambda_{2,3} = \frac{1}{2}\left(-1 \pm \sqrt{1-4}\right)$
$= \frac{1}{2}\left(-1 \pm \sqrt{3}\, i\right)$.

(c) $A = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Because this is a triangular matrix,
$\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 3$, $\lambda_4 = 1$.