HW #9 - Solutions

#11.1 $S = 8I_{6\times6}$, $T = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}$

To compute $\|S^{-1}T\|_\infty$ we find: $S^{-1} = \frac{1}{8}I^{-1} = \frac{1}{8}I$

$$S^{-1}T = \frac{1}{8}IT = \frac{1}{8}T$$

and $\|S^{-1}T\|_\infty = \|\frac{1}{8}T\|_\infty = \frac{1}{8}\|T\|_\infty = \frac{1}{8}(3) = \frac{3}{8} = 0.375.$

#11.3 $A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 6 & 2 \\ 1 & 1 & -5 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$, $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Notes: 1. A is strictly row diagonally dominant so both iterations show converge to the exact sol'n

2. The exact solution is $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Jacobi Method: $S = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -5 \end{bmatrix}$, $T = S - A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -2 \\ -1 & -1 & 0 \end{bmatrix}$.

$$Sx^{(1)} = b + Tx^{(0)} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} \Longrightarrow x^{(1)} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$Sx^{(2)} = b + Tx^{(1)} = \begin{bmatrix} 7/2 \\ -11/2 \\ -5 \end{bmatrix} \Longrightarrow x^{(2)} = \begin{bmatrix} 0.8750 \\ -0.9167 \\ 1 \end{bmatrix}$$

$$Sx^{(3)} = b + Tx^{(2)} = \begin{bmatrix} 3.9167 \\ -5.8750 \\ -4.9583 \end{bmatrix} \Longrightarrow x^{(3)} = \begin{bmatrix} 0.9792 \\ -0.9792 \\ 0.9917 \end{bmatrix}$$

$$Sx^{(4)} = b + Tx^{(3)} = \begin{bmatrix} 3.9308 \\ -5.9625 \\ -5 \end{bmatrix} \Longrightarrow x^{(4)} = \begin{bmatrix} 0.9927 \\ -0.9938 \\ 1 \end{bmatrix}. \quad x - x^{(4)} = \begin{bmatrix} 0.0073 \\ -0.0062 \\ 0 \end{bmatrix}$$

Gauss-Seidel Method: $S = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 6 & 0 \\ 1 & 1 & -5 \end{bmatrix}$, $T = S - A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$$Sx^{(1)} = b + Tx^{(0)} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} \Longrightarrow x^{(1)} = \begin{bmatrix} 0.5 \\ -0.5833 \\ 0.9833 \end{bmatrix}$$

$$Sx^{(2)} = b + Tx^{(1)} = \begin{bmatrix} 3.5667 \\ -4.9667 \\ -5 \end{bmatrix} \Longrightarrow x^{(2)} = \begin{bmatrix} 0.8917 \\ -0.9764 \\ 0.9831 \end{bmatrix}$$

$$Sx^{(3)} = b + Tx^{(2)} = \begin{bmatrix} 3.9594 \\ -4.9661 \\ -5 \end{bmatrix} \Longrightarrow x^{(3)} = \begin{bmatrix} 0.9899 \\ -0.9927 \\ 0.9994 \end{bmatrix}$$

$$Sx^{(4)} = b + Tx^{(3)} = \begin{bmatrix} 3.9921 \\ -4.9989 \\ -5 \end{bmatrix} \Longrightarrow x^{(4)} = \begin{bmatrix} 0.9980 \\ -0.9995 \\ 0.9997 \end{bmatrix} \qquad x - x^{(4)} = \begin{bmatrix} 0.0020 \\ -0.0005 \\ 0.0003 \end{bmatrix}$$

#11.4. $A = \begin{bmatrix} 2 & -1 & 0 \\ -6 & 9 & -2 \\ 8 & -18 & 8 \end{bmatrix}$ is not strictly diagonally row dominant so there is no easy answer using this criteria.

But, for Gauss-Seidel iterations, $S = \begin{bmatrix} 2 & 0 & 0 \\ -6 & 9 & 0 \\ 8 & -18 & 8 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

We can find (w/MATLAB) that $S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 1/9 & 0 \\ 1/4 & 1/4 & 1/8 \end{bmatrix}$ and $S^{-1}T = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/3 & 2/9 \\ 0 & 1/4 & 1/2 \end{bmatrix}$

Thus $\|S^{-1}T\|_\infty = \max\left(\frac{1}{2}, \frac{5}{9}, \frac{3}{4}\right) = \frac{3}{4} < 1$ so that Gauss-Seidel iterations will converge to the exact solution.

Note that for Jacobi's Method: $S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 0 & 2 \\ -8 & 18 & 0 \end{bmatrix}$

Here $S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}$ and $S^{-1}T = \begin{bmatrix} 0 & 1/2 & 0 \\ 2/3 & 0 & 2/9 \\ -1 & 9/4 & 0 \end{bmatrix}$

[Now $\|S^{-1}T\|_\infty = \max\left(\frac{1}{2}, \frac{8}{9}, \frac{13}{4}\right) = \frac{13}{4} > 1$, and we cannot say anything about the convergence of Jacobi's Method].

#11.7. Claim: $Sx^{new} = b + Tx^{old}$ is equivalent to $x^{new} = x^{old} + S^{-1}(b - x^{old})$.

Proof:
$$\begin{aligned}
Sx^{new} = b + Tx^{old} &\iff Sx^{new} = b + (S-A)x^{old} && (T = S - A) \\
&\iff Sx^{new} = Sx^{old} + b - Ax^{old}. \\
&\iff S^{-1}Sx^{new} = S^{-1}(Sx^{old} + b - Ax^{old}) && (S \text{ non-sing.}) \\
&\iff x^{new} = S^{-1}Sx^{old} + S^{-1}(b - Ax^{old}) && (S^{-1}S = I) \\
&\iff x^{new} = x^{old} + S^{-1}(b - Ax^{old}).
\end{aligned}$$

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